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High-Temperature Differentiability of Lattice Gibbs States by Dobrushin Uniqueness Techniques

Chetan Prakash[1]

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We establish conditions for the differentiability, to any order, of the Gibbs states of classical lattice systems with arbitrary compact single-spin space and with interactions in the Dobrushin uniqueness region. The derivatives are expressed as series expansions and are shown to be continuous on the uniqueness region. We also provide a procedure for estimating the size of the derivatives. These results verify a conjecture of L. Gross and extend his results in "Absence of second-order phase transitions in the Dobrushin uniqueness region," *Journal of Statistical Physics* **25**(1):57–72 (1981). The techniques of this paper are based on those employed by Gross.

KEY WORDS: Classical lattice spin systems; Dobrushin uniqueness theorem; differentiability of pressure.

1. INTRODUCTION

1.1. Discussion of Results

The purpose of this paper is to investigate the extent to which uniqueness of the Gibbs state of a lattice system implies differentiability properties of the pressure. In this section we discuss some of the earlier results in that direction and state our main theorem. In the next we establish background and notation.

In 1968, R. L. Dobrushin[(1)] demonstrated the uniqueness of Gibbs states for lattice models at high temperature. Specifically, there is a Banach space $\mathscr{P}$ of interactions and a neighborhood $\mathscr{D}$ of the origin in this space, such that to each interaction in $\mathscr{D}$ there corresponds a unique Gibbs state.

[1] Department of Mathematics, University of California at Irvine, Irvine, California.

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Both the single-spin space and the set of interactions for which Dobrushin's theorem holds are quite general: the spin space is compact metrizable but otherwise arbitrary, translational invariance is *not* assumed, and the interactions involved are many bodied and have long range [see Eqs. (1.13) and (1.14): $\mathscr{P} = \mathscr{P}_2$]. B. Simon has shown[(2)] that Dobrushin's theorem is, in a sense, quite strong: for any $\epsilon > 0$, there are interactions which are within a $\mathscr{P}$ distance ϵ of $\mathscr{D}$ and which possess multiple Gibbs states.

The Dobrushin uniqueness region $\mathscr{D}$ is, in fact, an open neighborhood of the origin in $\mathscr{P}$, as shown by L. Gross.[(3)] In the translationally invariant context, where the pressure $P(\phi)$ may be defined, this leads immediately to a differentiability result. Namely, the pressure is "Gateaux differentiable in $\mathscr{P}$ directions" at each interaction ϕ in $\mathscr{D}$. By this we mean that for each ϕ in $\mathscr{D}$ and ψ in $\mathscr{P}$, the function $u \to P(\phi + u\psi) : R^1 \to R^1$ is differentiable at $u = 0$, in which case we write

$$\partial_\psi P(\phi) = \frac{d}{du} P(\phi + u\psi)|_{u=0}$$

The Gateaux derivative is also called the "functional" or "directional" derivative. Thus we assert that the pressure is differentiable in parameters occurring linearly in the interactions. This follows from the openness of $\mathscr{D}$, the convexity of $P(\phi)$, and the fact that every Gibbs state corresponding to an interaction may be represented by a "tangent functional" to the pressure at that interaction. See Ref. 4, p. 96 for details.

The properties of convex functions lead, moreover, to the fact that the pressure is *continuously* Gateaux differentiable on $\mathscr{D}$, i.e., that the function $\phi \to \partial_\psi P(\phi)$ is, for fixed $\psi \in \mathscr{P}$, continuous on $\mathscr{D}$. If σ_ϕ is the unique Gibbs state corresponding to ϕ in $\mathscr{D}$, the derivative may be written

$$\partial_\psi P(\phi) = \sigma_\phi(A_\psi)$$

where A_ψ is an observable (i.e., a continuous function on the space of spin configurations) which is linearly associated with ψ [see (1.17)], and where $\sigma_\phi(A_\psi)$ is the expectation of A_ψ with respect to the probability measure σ_ϕ.

The main result of Gross in Ref. 3 is that the pressure is actually twice continuously Gateaux differentiable on $\mathscr{D}$, in $\mathscr{P}$ directions. This obviates "second-order" phase transitions. He achieved this by showing that there is a dense subspace, C^1, of observables such that the function $\phi \to \sigma_\phi(f)$ is once continuously Gateaux differentiable on $\mathscr{D}$, for each f in C^1 (note: if $\psi \in \mathscr{P}$ then $A_\psi \in C^1$; see Section 2.5).

The main result of this paper (Theorem 5.1) extends that of Gross above as follows: We identify a decreasing sequence $[\mathscr{P}_N]$ of interaction spaces ($N \geqslant 2$, $\mathscr{P} = \mathscr{P}_2$) and a decreasing sequence $[C^N]$, $N \geqslant 1$ of dense subspaces of observables such that, for each $f \in C^N$, the function $\phi \to \sigma_\phi(f)$ on $\mathscr{D} \cap \mathscr{P}_{N+1}$ is N-times continuously Gateaux differentiable (in $\mathscr{P}_{N+1}$

directions). The techniques used are based on those of Gross, who, in turn, extended the techniques used by Lanford[4] and Vasershtein[5] in their proofs of Dobrushin's uniqueness theorem.

We also provide series expansions for the derivatives (as in Ref. 3 for $N = 1$) as well as a procedure for estimating their size. These expansions seem to bear no resemblance to the customary series involving truncated correlations. As pointed out in Ref. 3, they may, however, be more useful for computations. However, H. Künsch[6] has recently shown that the first derivative of the Gibbs state is in fact equal to the usual series of covariances.

As a corollary to our main result, it follows that, in the translationally invariant context, the pressure is N-times continuously differentiable in $\mathscr{D} \cap \mathscr{P}_N$, $N \geqslant 2$.

We note here that the intersection of all the $\mathscr{P}_N$ spaces is still quite large. It contains, for example, all finite-range interactions. For a sufficient condition for interactions to lie in $\mathscr{D}$, see (1.18).

There are a number of problems immediately related to our main result which remain undecided. Some of these ought to succumb to the methods we have used. Firstly, it remains to investigate whether the pressure is *not* N times continuously differentiable on $\mathscr{D} \cap \mathscr{P}_{N-1}$. Secondly, we have not been able to establish the equality between the derivatives and the corresponding series of truncated correlations for arbitrary N. Thirdly, in Ref. 7 Gross demonstrated that the averaged two-point truncated correlations decay in the same weighted summability sense as the potential, when the latter lies in a suitable open subset of $\mathscr{D}$. (Improved estimates have recently been obtained by H. Künsch[6] and H. Föllmer.[8]) We conjecture that a similar result holds in $\mathscr{D} \cap \mathscr{P}_N$, $N = 3, 4, \ldots$. (For an extension of the main results of Ref. 7 in the direction of continuum systems see Ref. 9, which also provides an extension of Dobrushin's theorem to this situation.) Finally, we have not identified the exact domain of real analyticity of the pressure.

A number of results on *complex* analyticity of the pressure are extant in the literature. Using Dobrushin uniqueness techniques, R. B. Israel[10] established complex analyticity, at high temperature, in any finite number of directions. The real part of his space of (complex) interactions is contained in the intersection of our $\mathscr{P}_N$, but from the point of view of range and many-bodiedness is still quite general. Results on analyticity not using Dobrushin techniques usually exert more stringent restrictions, for example, on the cardinality of the single-spin space. G. Gallavotti and S. Miracle-Sole[11] have demonstrated analyticity at high temperature (or low activity) for a wider class of potentials than ours (the space $\mathscr{P}_1$ of "supersummable" interactions), though their single-spin space is restricted to two points. Their techniques do not seem to admit generalization to infinite cardinality. Links between high-temperature correlation functions and analyticity have been explored by Duneau, Iagolnitzer, and Souillard[12–15] and Holley and Stroock.[16]

1.2. The Mathematical Setting

General references for this section are Refs. 17 and 18.

Let L be a countably infinite set with some fixed enumeration $\{a_i\}$. Let X be a compact metrizable space and associate with each $a \in L$ a copy X_a of X. For any $\Lambda \subset L$, define $\Omega_\Lambda = \times_{a \in \Lambda} X_a$ and $\Omega = \Omega_L$. Ω_Λ is the space of configurations inside Λ. We shall call a function in $C(\Omega)$ that depends upon only finitely many coordinates a (continuous) cylinder function. Since Ω is compact, the cylinder functions are dense in $C(\Omega)$.

In all that follows, the notation "$\Lambda \subset L$" will be taken to refer to a *finite* subset of Λ of L. Also, we shall write "s_p" for the (a_p)th coordinate of $s \in \Omega$, and often write "$j \in L$," "$\Lambda \cup j$," etc. instead of $a_j \in L$, $\Lambda_\cup \{a_j\}$, respectively. We shall sometimes speak of a cylinder function as being "based upon p" if it is independent of all jth coordinates with $j > p$.

Let $M = \{\mu_j(dx \mid s)\}_{j \in L, s \in \Omega}$ be a system of "conditional measures," i.e.,

(i) $\mu_j(\cdot \mid s)$ is a probability measure on X_j, for each $s \in \Omega$.

(ii) $\mu_j(\cdot \mid s)$ is independent of s_j.

(iii) For any continuous function g on X_j, $\int_{X_j} \mu_j(dx \mid s)\, g(x)$ is in $C(\Omega)$. (The μ_j will arise as single-site Gibbs ensembles.) Following Lanford, define the operators ζ_j on $C(\Omega)$ by

$$\zeta_j f(s) = \int_{X_j} \mu_j(dx \mid s)\, f(x_\vee \hat{s}) \tag{1.1}$$

where $\hat{s} = s|_{L-j}$ and $x_\vee \hat{s}$ is the configuration obtained from s by replacing its jth component with x. That $\zeta_j f$ is a continuous function follows from (iii) above and (ii) shows that it is independent of s_j. It is a contraction (i.e., $|\zeta_j f|_\infty \leqslant |f|_\infty$, where $|\cdot|_\infty$ is the supremum norm) since $\mu_j(\cdot|s)$ is a probability measure.

For $k \leqslant p < \infty$ define

$$T_{k,p} f = \zeta_k \cdot \zeta_{k+1} \cdots \xi_p f \tag{1.2}$$

As a product of contractions, $T_{k,p}$ is a contraction itself. But note that the function $T_{k,p}f(s)$ is, in general, only independent of s_k. The function $\zeta_p f$, for example, is independent of s_p, but applying ζ_{p-1} to it introduces a dependence on s_p by virtue of the dependence of $\mu_{p-1}(\cdot|s)$ on s_p.

Consider the sequence $\{T_{1,p}f\}_{p=1}^\infty$, for fixed $f \in C(\Omega)$. If f is a cylinder function based on q, then it is clear from (1.1) that $\zeta_j f = f$ whenever $j > q$.

Thus the sequence $\{T_{1,p}f\}$ becomes stationary as soon as $p > q$ (in which case $T_{1,p}f = T_{1,q}f$) and we may define $Tf = \lim_{p\to\infty} T_{1,p}f$ [where the limit is in the uniform, or $C(\Omega)$, norm]. Since the $T_{1,p}$ are uniformly bounded and the cylinder functions are dense in $C(\Omega)$, it follows that the limit Tf exists for all $f \in C(\Omega)$. The same arguments hold for $\{T_{k,p}f\}_{p=1}^{\infty}$, in which instance we write $T_{k,\infty} f = \text{Lim}_{p\to\infty} T_{k,p}f$.

Define, for $k \in L$,

$$R_{k,j} = \tfrac{1}{2}\sup\{\| \mu_j(\cdot\,|s) - \mu_j(\cdot\,|t)\| : s = t \text{ off } k\} \tag{1.3}$$

where the norm is that of total variation. Let

$$\alpha_1 = \sup_j \sum_k R_{k,j} \tag{1.4}$$

We are now in a position to state the following:

Dobrushin's Uniqueness Theorem. If $\alpha_1 < 1$, there is at most one probability measure σ on the configuration space Ω such that $\sigma(\zeta_j f) = \sigma(f)$ for all $j \in L$ and $f \in C(\Omega)$.

For a proof, see Corollary 3.3 of Ref. 7. Vasershtein's method of proof[5] gives us a neat formula for constructing the Gibbs state, namely, if σ is a measure as above,

$$\sigma(f) = \lim_{n\to\infty} T^n f, \qquad f \in C(\Omega) \tag{1.5}$$

That is, $T^n f(s)$ tends, uniformly in s, to the constant $\sigma(f)$.

Crucial to the method of proof is the identification of a dense subset C^1 of functions on which (1.5) is first established. This space is defined as follows:

For each $j \in L$ and $f \in C(\Omega)$, define

$$D_j(f) = \sup\{|f(s) - f(t)| : s = t \text{ off } j\}$$
$$|f|_1 = \sum_j D_j(f) \tag{1.6}$$

Then $C^1 = \{f \in C(\Omega) : |f|_1 < \infty\}$. Every cylinder function is contained in C^1, so that C^1 is dense in $C(\Omega)$. The proof of Dobrushin's theorem we use relies on the fact that for $\alpha_1 < 1$, $|T^n f|_1 \leqslant \alpha_1 |f|_1$.

In statistical mechanics we define the system of conditional measures by means of an interaction. This is a continuous function ϕ on $\bigcup_{\Lambda\subset L}\Omega_\Lambda$ (again, the Λ are *finite*) whose value on Ω_Λ represents the many-body interaction energy between the spins located in Λ. The many-body energy is denoted $\phi(s\,|\,\Lambda)$ (for a configuration s in Ω) and we require that $\phi(s\,|\,\varnothing) = 0$.

Thus the energy of any configuration inside $\Gamma \subset L$ is given by $\bigcup_\Gamma(s) = \sum_{\Lambda\subset\Gamma}\phi(s\,|\,\Lambda)$. Evidently the contribution of any single site j in the lattice to the total interaction energy of some configuration s in Ω is given by

$$-k_{\phi,j}(s) = \sum_{\substack{\Lambda \subset L \\ j \in \Lambda}} \phi(s \mid \Lambda) \tag{1.7}$$

(assuming the series converges). We require that

$$\|\phi\|_1 = \sup_{j \in L} \sum_{\substack{\Lambda \subset L \\ j \in \Lambda}} \sup_{s \in \Omega} |\phi(s \mid \Lambda)| < \infty \tag{1.8}$$

so that $k_{\phi,j}(s)$ is finite and, by uniform convergence, is a continuous function on Ω. Let ν be some bounded real Borel measure (the "*a priori* single-spin" measure) and define the single-spin Gibbs ensemble at j (with "boundary conditions" $\hat{s} = s|_{L-j}$) by

$$\mu_j^\phi(dx \mid s) = Z_j^\phi(s)^{-1} \exp\left[k_{\phi,j}(x_\vee \hat{s}) \right] \nu(dx) \tag{1.9}$$

where $Z_j^\phi(s)$ is the normalization. It is easily checked that $M = \{ \mu_j^\phi \}$ is a system of conditional measures. In this case we write ζ_j^ϕ in place of ζ_j, $T_{\phi,k,p}$ for $T_{k,p}$ and $\alpha_1(\phi)$ for α_1.

We proceed now to define Gibbs states corresponding to ϕ. Put

$$W_\Lambda^\phi(s) = \sum_{\substack{\Gamma \subset L \\ \Gamma \cap \Lambda \neq \varnothing}} \phi(s \mid \Gamma) \tag{1.10}$$

(we have $|W_\Lambda^\phi| \leqslant |\Lambda| \, \|\phi\|_1$, where $|\Lambda|$ is the cardinality of Λ). $W_\Lambda(s)$ is to be interpreted as the energy of the configuration $s|_\Lambda$ inside Λ, plus the energy of interaction between $s|_\Lambda$ and the "boundary" spins $s|_{L-\Lambda}$ outside Λ. Write

$$\zeta_\Lambda^\phi f(s) = Z_\Lambda(s)^{-1} \int_{\Omega_\Lambda} \nu^\Lambda(dx) \, e^{-W_\Lambda^\phi(x_\vee \hat{s})} f(x_\vee \hat{s}) \tag{1.11}$$

where now $\hat{s} = s|_{L=\Lambda}$ and $\nu^\Lambda = \bigotimes_\Lambda \nu$. A Gibbs state corresponding to ϕ with $\|\phi\|_1 < \infty$ is any probability measure σ on Ω satisfying

$$\sigma(\zeta_\Lambda^\phi f) = \sigma(f) \qquad \text{for all} \quad \Lambda \subset L, \quad f \in C(\Omega) \tag{1.12}$$

That is, the Gibbs ensemble ζ_Λ^ϕ is just the conditional expectation corresponding to σ, given the configuration of spins outside Λ. If we think of the ζ_Λ as probability measures on $C(\Omega_\Lambda)$, then it is easily seen by a compactness argument that the set of weak limits of these measures as $\Lambda \uparrow L$ is nonempty and consists of Gibbs states. Dobrushin's theorem asserts that if $\alpha_1(\phi) < 1$ there exists a unique Gibbs state.

Following Gross we now define the spaces $\mathscr{P}_N$ alluded to in Section 1.1. For any integer N and interaction ϕ, put

$$\|\phi\|_N = \sum_{\substack{\Lambda \subset L \\ j \in \Lambda}} |\Lambda|^{N-1} \sup_{s \in \Omega} |\phi(s \mid \Lambda)| \tag{1.13}$$

and define

$$\mathscr{P}_N = \{\phi : \|\phi\|_N < \infty\} \quad \text{and} \quad \mathscr{D} = \{\phi \in P_2 : \alpha_1(\phi) < 1\} \tag{1.14}$$

(we choose $\mathscr{P}_2$ rather than $\mathscr{P}_1$ because $\mathscr{D}$ is not open in $\mathscr{P}_1$). If $L = Z^d$, the corresponding spaces of translationally covariant interactions are denoted by $\hat{\mathscr{P}}_N$. Each $\mathscr{P}_N$ is a separable Banach space.

An aside: It is possible to define the pressure on $\hat{\mathscr{P}}_0$. However this space is quite pathological from a physical viewpoint. For example, M. E. Fisher constructed in Ref. 19 models whose interactions lie in $\hat{\mathscr{P}}_0$ and whose pressure is discontinuous as a function of density, implied by linear segments on the $P - \mu$ isotherms. This phenomenon has never been observed in the laboratory and it is known[20,21] that such a lack of strict convexity cannot obtain in $\mathscr{P}_1$ (and therefore in $\mathscr{P}_N$, $N \geqslant 1$). In Fisher's example two physically distinct interactions share the same Gibbs state. Indeed, in Ref. 22 Israel found that there exists a dense set of interactions in $\hat{\mathscr{P}}_0$, each of which has uncountably many Gibbs states. We shall not dwell upon $\hat{\mathscr{P}}_0$ any further.

Let us now define the pressure on $\hat{\mathscr{P}}_1$ with $L = Z^d$. With W_Λ^ϕ as in (1.10), put for each $s \in \Omega$

$$P_\Lambda(\phi)(s) = \frac{1}{|\Lambda|} \log \int_{\Omega_\Lambda} \nu^\Lambda(dx) \exp\left[- W_\Lambda^\phi(x_\vee \hat{s}) \right] \tag{1.15}$$

The "thermodynamic limit" of $P_\Lambda(\phi)(s)$ as Λ approaches Z^d through a sequence of sets (with suitably tame surfaces) exists uniformly in, and independent of, the boundary conditions s. This limit is the pressure $P(\phi)$. The P_Λ and P are convex, continuous functions on $\hat{\mathscr{P}}_1$ (and therefore on $\hat{\mathscr{P}}_N$, $N \geqslant 1$). In fact, they are Lip 1 functions:

$$|P(\phi) - P(\phi')| \leqslant \|\phi - \phi'\|_1 \leqslant \|\varphi - \varphi'\|_N, \qquad N \geqslant 1 \tag{1.16}$$

In Section 1.1 we alluded to the fact that a unique Gibbs state σ corresponding to the interaction ϕ is representable as a tangent functional to the pressure. By this we mean that, with 0 the origin of Z^d,

$$P(\phi + \psi) \geqslant P(\phi) + \sigma(A_\psi), \qquad \text{where} \quad A_\psi(s) = - \sum_{\Lambda : 0 \in \Lambda} |\Lambda|^{-1} \psi(s \mid \Lambda) \tag{1.17}$$

A sufficient condition for ϕ to be in $\mathscr{D}$ was given by B. Simon in Ref. 2 namely,

$$\sup_j \sum_{\Lambda : j \in \Lambda} (|\Lambda| - 1) \phi(\cdot \mid \Lambda)|_\infty < 1 \tag{1.18}$$

For example, if $X = S^{n-1}$, the unit sphere in R^n, and ϕ is a purely pair interaction, with $\phi(s \mid \{a, b\}) = -J_{a,b} \mathbf{s}_a \mathbf{s}_b$ (so that we have the "n-vector" model) and $\sum_b |J_{0,b}| < 1$, there is a unique Gibbs state and the pressure is infinitely differentiable. In general, interactions in $\mathscr{P}_1$ satisfying (1.18) but having vanishing M-body interactions for all M larger than some number will be in $\mathscr{D}_\cap \mathscr{P}_N$ for all N and therefore possess infinitely differentiable Gibbs states.

We proceed now to informally differentiate the Vasershtein equation (1.5) for the Gibbs state expectation. We write

$$\begin{aligned} \partial_\psi T_\phi f &= \frac{d}{du} T_{\phi + u\psi} f|_{u=0} \\ \partial_{\psi_1,\psi} T_\phi f &= \frac{d}{du} (\partial_\psi T)_{\phi + u\psi_1} f|_{u=0} \end{aligned} \tag{1.19}$$

and so on. Since

$$T_{\phi+u\psi} f - T_\phi^n f = \sum_{k=1}^{n} T_{\phi+u\psi}^{n-k} \cdot (T_{\phi+u\psi} - T_\phi) T_\phi^{k-1} f \tag{1.20}$$

we have

$$\partial_\psi \sigma_\phi(f) = \lim_{n\to\infty} \sum_{k=1}^{n} T_\phi^{n-k} \cdot \partial_\psi T_\phi \cdot T_\phi^{k-1} f \tag{1.21}$$

Brazenly performing this limit and using (1.5) again,

$$\partial_\psi \sigma_\phi(f) = \sum_{k=1}^{\infty} \sigma_\phi\left(\partial_\psi T_\phi \cdot T_\phi^{k-1} f\right) \tag{1.22}$$

This formula was rigorously established by Gross[3] under the conditions that $\phi \in \mathscr{D}$, $\psi \in \mathscr{P}_2$ and $f \in C^1$. He also pointed out that it can be written in the form

$$\partial_\psi \sigma_\phi(f) = \sigma_\phi\left(\partial_\psi T_\phi \cdot (I - T_\phi)^{-1}[f]\right) \tag{1.23}$$

Here $[f]$ is the equivalence class $\{f + c : c \text{ a constant function}\}$. The C^1 seminorm defined in (1.6) has as its kernel the constant functions. Thus $\tilde{C}^1 = (C^1/\text{constants})$ is a Banach space; since $I - T_\phi$ annihilates constants it may be interpreted as an operator on $\tilde{C}^I$, with inverse given by

$$S_\phi[f] = (I - T_\phi)^{-1}[f] = \sum_{k=1}^{\infty} \left[T_\phi^{k-1} f\right] \tag{1.24}$$

Similarly, $\partial_\psi T_\phi$ acting on C^1 annihilates constants; Gross showed that $\partial_\psi T_\phi$ takes C^1 into $C(\Omega)$ and so may be interpreted as an operator from $\tilde{C}^1$ to $C(\Omega)$.

Space limitations prevent us from providing the heuristics for establishing higher derivatives [though it is clear how to proceed informally—see (5.2) below]. Heuristics are given in Ref. 23, Section 1.3. Here we note that we need to identify the spaces $\mathscr{P}_N$ and C^N (of interactions and functions, respectively) on which the informal expressions for the Nth derivative are well defined and converge. If the topologies of C^N are defined by means of seminorms, as for C^1, then we at least want seminorms for which the operators ζ_j, T, etc. are bounded. We follow this clue in the next section, where we define C^N seminorms in terms of "Nth-order oscillations" by analogy to the definition of the C^1 seminorm in terms of "first-order" oscillations. In Section 3 we show that the operators T_ϕ are bounded on C^N for ϕ in $\mathscr{D}_\cap \mathscr{P}_{N+1}$. In Section 4 we study the derivatives of the $\zeta_j^\phi f$ and establish their boundedness on C^N and continuity in ϕ. In the final section we state and prove the main theorem, giving estimates of the first two derivatives of the Gibbs state.

Space limitations also prevent us from giving detailed proofs in the first four sections, though we have tried to indicate the main ideas. This paper is a condensation of Ref. 23, where details may be found.

2. THE SPACES C^N AND THE BOUNDEDNESS OF THE ζ_j^ϕ ON C^N

2.1. Nth-Order Differences

For $\Lambda \subset L$ put

$$\Omega'_\Lambda = \{(s,t) \in \Omega^2 : s = t \text{ off } \Lambda\} \tag{2.1}$$

(the topology of Ω'_Λ is given by its natural association to $\Omega_\Lambda \times \Omega_\Lambda \times \Omega_{L-\Lambda}$). For any $\Lambda \subset L$ and $(s,t) \in \Omega'_\Lambda$, consider the set

$$\sum(s,t) = \{w \in \Omega : w_i = \text{either } s_i \text{ or } t_i,\ i \in L\} \tag{2.2}$$

Then $\sum(s,t)$ is the set of at most $2^{|\Lambda|}$ configurations in Ω which may be constructed from the pair (s,t) by changing some components of s into those of t. We may symbolize this process as follows: For $w \in \sum(s,t)$ and $\Gamma \subset \Lambda$, let $w_\Gamma \in \Omega$ be given by

$$(w_\Gamma)_i = \begin{cases} t_i & \text{if } i \in \Gamma \text{ and } w_i = s_i \\ s_i & \text{if } i \in \Gamma \text{ and } w_i = t_i. \\ w_i & \text{if } i \notin \Gamma \end{cases} \tag{2.3}$$

Then w_Γ is also in $\sum(s,t)$ and is obtained from w by "switching" the appropriate components of w. Note that if $w \in \sum(s,t)$, then $w = s_\Gamma$ for some $\Gamma \subset \Lambda$. It is evident that if $(s,t) \in \Omega'_\Lambda$, then for any $u \in \sum(s,t)$ we have $(u, u_\Lambda) \in \Omega'_\Lambda$ and moreover that $\sum(s,t) = \sum(u, u_\Lambda)$.

Let $F: \Omega \to Y$, where Y is an additive Abelian group (pertinent examples are Y = real line or Y = finite signed measures, on some X_j, indexed by Ω, such as $\mu_j(dx \mid s)$ of Section 1). Given $\Lambda \subset L$, define the ($|\Lambda|$th-order) *difference of F on Λ*, denoted F^{Λ}, by

$$F^{\Lambda}:\Omega'_{\Lambda}\rightarrow Y: F^{\Lambda}(s,t)=\sum_{\Gamma\subset\Lambda}(-1)^{|\Gamma|}F(s_{\Gamma}) \tag{2.4}$$

where s_{Γ} is to be interpreted with reference to (s,t) as above. Note that $F^{\phi}(s)=F(s)$.

Differences up to second order are as follows: let $a,b\in L$, $u\in\Omega_{L-a}$, $v\in\Omega_{L-\{a,b\}}$. Then

$$F^{\phi}:\Omega\rightarrow Y: F^{\phi}(s)=F(s)$$

$$\begin{aligned} F^{\{a\}}&:\Omega'_{\{a\}}\rightarrow Y: F^{\{a\}}(s_{a\vee}u, t_{a\vee}u)=F(s_{a\vee}u)-F(t_{a\vee}u) \\ F^{\{a,b\}}&:\Omega'_{\{a,b\}}\rightarrow Y: F^{\{a,b\}}(s_{a\vee}s_{b\vee}v, t_{a\vee}t_{b\vee}v) \\ &=F(s_{a\vee}s_{b\vee}v)-F(t_{a\vee}s_{b\vee}v)-F(s_{a\vee}t_{b\vee}v)+F(t_{a\vee}t_{b\vee}v)\end{aligned} \tag{2.5}$$

Various properties of these differences appear in Section 2.3. We note here that the concept is really one of "differences of differences" and accordingly we may generalize (2.4) as follows.

Suppose that $\Upsilon\subset L$ and $G:\Omega'_{\Upsilon}\rightarrow Y$, Y an additive Abelian group. For any $\Lambda\subset L-\Upsilon$, define the difference of G on Λ as

$$G^{\Lambda}:\Omega'_{\Lambda\cup\Upsilon}\rightarrow Y: G^{\Lambda}(s,t)=\sum_{\Gamma\subset\Lambda}(-1)^{|\Gamma|}G(s_{\Gamma}, s_{(\Gamma\cup\Upsilon)}) \tag{2.6}$$

2.2. Definition of the Seminorms $D_{\Lambda}, |\cdot|_N$

Let $\Lambda\subset L$ and $F\in C(\Omega)$. Put

$$D_{\Lambda}(F)=\sup_{\Omega'_{\Lambda}}|F^{\Lambda}(\cdot)| \tag{2.7}$$

$$|F|_N=\sum_{\substack{\Lambda\subset L\\|\Lambda|=N}}D_{\Lambda}(F) \tag{2.8}$$

We shall call $|\cdot|_N$ the Nth *seminorm*.

More generally, suppose that $G:\Omega'_{\Upsilon}\rightarrow Y$ and $\Lambda\subset L-\Upsilon$. Put

$$D_{\Lambda}(G)=\sup_{\Omega'_{\Lambda\cup\Upsilon}}|G^{\Lambda}(\cdot)| \tag{2.9}$$

Of interest to us is when $G=F^{\Upsilon}$, $F\in C(\Omega)$, in which instance $D_{\Lambda}(F^{\Upsilon}) = D_{\Lambda\cup\Upsilon}(F)$ (see Section 2.4).

2.3. Properties of Nth Order Differences

We will consider only functions $F:\Omega\rightarrow Y$, as in (2.4). Extensions to the situation of (2.6) are trivial.

Fix $\Lambda\subset L$ and $(s,t)\in\Omega'_{\Lambda}$. We have the following:

(a) For $\Gamma, \Gamma' \subset \Lambda$ and $w \in \sum(s,t)$,

$$(w_\Gamma)_{\Gamma'} = w_{\Gamma\Delta\Gamma'}, \qquad \text{where} \quad \Gamma\Delta\Gamma' = (\Gamma - \Gamma') \cup (\Gamma' - \Gamma) \tag{2.10}$$

(b) We have seen in Section 2.1 that to $w \in \sum(s,t)$ there corresponds a $\Gamma \subset \Lambda$ such that $w = s_\Gamma$. Also, $(w, w_\Lambda) \in \Omega'_\Lambda$. We have, in that case

$$F^\Lambda(w, w_\Lambda) = (-1)^{|\Gamma|} F^\Lambda(s,t) \tag{2.11}$$

(c) Let $\Upsilon \subset L - \Lambda$. Then

$$F^{\Upsilon \cup \Lambda} = (F^\Upsilon)^\Lambda \tag{2.12}$$

as in (2.6) with $G = F^\Upsilon$. This is a recursion formula for the differences. In particular, suppose $a \in \Lambda$. Let $x = s_a$, $y = t_a$, $\hat{s} = s|_{L-a}$ and $\hat{t} = t|_{L-a}$. Then

$$F^\Lambda(s,t) = F^\Lambda(x_\vee \hat{s}, y_\vee \hat{t}) = F^{\Lambda - a}(x_\vee \hat{s}, x_\vee \hat{t}) - F^{\Lambda - a}(y_\vee \hat{s}, y_\vee \hat{t}) \tag{2.13}$$

(d) If $s_i = t_i$ for some $i \in \Lambda$, $F^\Lambda(s,t) = 0$.

(e) If $F(s)$ is independent of s_j, $F^\Lambda \equiv 0$ whenever $j \in \Lambda$.

2.4. Remarks on the Seminorms $D_\Lambda, |\cdot|_N$

(a) If $F \in C(\Omega)$, (2.4) shows that $D_\Lambda(F) \leqslant 2^{|\Lambda|}|F|_\infty < \infty$. This and the linearity of $F \to F^\Lambda$ show that the D_Λ are indeed seminorms on $C(\Omega)$.

(b) Suppose $j \notin \Lambda$, $F \in C(\Omega)$ and there is a function ξ on Ω'_Λ such that $\xi(s,t)$ does not depend upon $s_j = t_j$ and such that it has the following property: for each $(s,t) \in \Omega'_\Lambda$ with $\hat{s} = s|_{L-j}$ and $\hat{t} = t|_{L-j}$,

$$\inf_{x \in X_j} F^\Lambda(x_\vee \hat{s}, x_\vee \hat{t}) \leqslant \xi(s,t) \leqslant \sup_{x \in X_j} F^\wedge(x_\vee \hat{s}, x_\vee \hat{t}) \tag{2.14}$$

Then

$$\sup_{\Omega'_\Lambda} |F^\Lambda - \xi| \leqslant D_{\Lambda \cup j}(F) \tag{2.15}$$

For,

$$\begin{aligned}|F^\Lambda(s,t) - \xi(s,t)| &\leqslant \sup_{x \in X_j} F^\Lambda(x_\vee \hat{s}, x_\vee \hat{t}) - \inf_{x \in X_j} F^\Lambda(x_\vee \hat{s}, x_\vee \hat{t}) \\ &\leqslant \sup_{x,y \in X_j} |F^\wedge(x_\vee \hat{s}, x_\vee \hat{t}) - F^\Lambda(y_\vee \hat{s}, y_\vee \hat{t})| \\ &\leqslant D_{\Lambda \cup j}(F)\end{aligned} \tag{2.16}$$

where we have used (2.13).

(c) If $F(s)$ is independent of s_j and $j \in \Lambda$, Property 2.3(e) shows that $D_\Lambda(F) = 0$. In consequence, if F is a continuous cylinder function, $|F|_M < \infty$ for all $M = 1, 2, \ldots$.

(d) Let $s, t \in \Omega$ and $F \in C(\Omega)$. If we change the coordinates of s into those of t one at a time, we see that $|F(s) - F(t)| \leqslant |F|_1$. Thus the subset of $C(\Omega)$ on which the "seminorm" $|\cdot|_1$ vanishes consists exactly of the constant functions.

2.5. Definition of the Spaces C^N; Their Connection with Spaces of Physical Interactions

For each natural number N we define C^N to be the topological vector space consisting of those $f \in C(\Omega)$ with $|f|_M < \infty$, $1 \leqslant M \leqslant N$, and with topology defined by this collection of seminorms.

I am indebted to John Reid of U.C.I. for pointing out to me the following:

Lemma. If $f \in C(\Omega)$, there exists a sequence $\{f_n\}$ of cylinder functions converging to f uniformly and such that $D_\Lambda(f_n) \leqslant D_\Lambda(f)$ for all $\Lambda \subset L$ and all n.

This may be seen as follows. Let $\{g_n\}$ be a sequence of cylinder functions tending to f uniformly and let Λ_n be finite subsets of L such that g_n depends only on coordinates in Λ_n for all n. For $s \in \Omega$, let $s_n = s \mid \Lambda_n$ and let $t_n \in \Omega_{L-\Lambda_n}$ be any choice of configurations. Then $f_n(s) = f(s_n \vee t_n)$ are the required cylinder functions. It follows from this lemma that

$$C^N = \big[f \in C(\Omega) : |f|_M < \infty,\ 1 \leqslant M \leqslant N, \text{ and there exists a sequence } \{f_n\} \text{ of cylinder functions such that } |f - f_n|_M \to 0,\ 1 \leqslant M \leqslant N \text{ and } |f - f_n|_\infty \to 0 \big] \tag{2.17}$$

As we have seen in Section 1, the functions in $C(\Omega)$ that we expect to arise out of successive differentiations of the pressure $P(\phi)$ are of the following two kinds, namely,

$$A_\phi(s) = -\sum_{\Gamma : j \in \Gamma} |\Gamma|^{-1} \phi(s \mid \Gamma) \tag{2.18}$$

and

$$k_\phi(s) = -\sum_{\Gamma : j \in \Gamma} \phi(s \mid \Gamma) \tag{2.19}$$

for some $j \in L$. By a standard computation employing Property 2.3(e) it may be shown that

$$|k_\phi|_N \leqslant 2^N \sum_{\Gamma \in j} \binom{|\Gamma|}{N} |\phi(\cdot \mid \Gamma)|_\infty \leqslant \frac{2^N}{N!} \|\phi\|_{N+1} \tag{2.20}$$

and similarly that

$$|A_\phi|_N \leqslant \frac{2^N}{N!} \|\phi\|_N \tag{2.21}$$

Thus we have that, for each nonnegative integer N, $\phi \to k_\phi : \mathscr{P}_{N+1} \to C^N$ and $\phi \to A_\phi : \mathscr{P}_N \to C^N$. Moreover, these maps are surjective for $N = 0$.[18]

2.6. Definition of $R_{\Lambda,j}$ and α_N

As in Section 1, define the probability measures

$$\mu_j^\phi(dx \mid s) = \nu_j(dx) \frac{\exp k_\phi(x_\vee \hat{s})}{Z(s)}, \qquad x \in X_j \tag{2.22}$$

for each $j \in L$ and $s \in \Omega$. If $F(s) = \mu_j^\phi(dx \mid s)$, write

$$\left(\mu_j^\phi\right)^\Lambda(dx \mid s,t) = F^\Lambda(s,t), \qquad \Lambda \subset L$$

Define

$$R_{\Lambda,j}(\phi) = \tfrac{1}{2}\sup\left\{ \left\|\left(\mu_j^\phi\right)^\Lambda(dx \mid s,t)\right\|_{\text{total variation}} : (s,t) \in \Omega'_\Lambda \right\} \tag{2.23}$$

Note that $R_{\Lambda,j} = 0$ if $j \in \Lambda$. Also, put

$$\alpha_N(\phi) = \sup_j \sum_{\substack{\Lambda \subset L \\ |\Lambda| = N}} R_{\Lambda,j}(\phi), \qquad N = 1, 2, \ldots \tag{2.24}$$

It will be seen in Section 4.2 that the $\alpha_N(\cdot)$ are finite and in fact continuous on $\mathscr{P}_{N+1}$.

We next present the fundamental estimate needed to show that T_ϕ is bounded on C^N. This generalizes the basic inequality of Lanford,[4] who found that $D_a(\zeta_a f) = 0$ and

$$D_a(\zeta_j f) \leqslant D_a(f) + R_{a,j} D_j(f), \qquad a \neq j \tag{2.25}$$

2.7. Theorem Estimating $D_\Lambda(\zeta_j^\phi f)$

Let $f \in C(\Omega)$, $j \in L$, and $\Lambda \subset L$. Then

$$D_\Lambda\left(\zeta_j^\phi f\right) \leqslant \begin{cases} 0, & j \in \Lambda \\ D_\Lambda(f) + \sum\limits_{\substack{\Gamma \subset \Lambda \\ \Gamma \neq \varnothing}} R_{\Gamma,j}(\phi) D_{(\Lambda - \Gamma) \cup j}(f), & j \notin \Lambda \end{cases} \tag{2.26}$$

(If $\Lambda = \varnothing$, we interpret the sum over Γ in the right-hand side to be 0.)

For the proof, the essential step is to establish, by induction on $|\Lambda|$, the following identity (where we suppress the symbols j and ϕ):

$$2^{|\Lambda|}(\zeta f)^\Lambda(s,t) = \sum_{\Gamma \subset \Lambda} \sum_{\Gamma' \subset \Lambda} (-1)^{|\Gamma'|} \int_{X_j} \mu^\Gamma\left[dx \mid s_{\Gamma'}, (s_{\Gamma'})_\Gamma\right]$$

$$\times f^{\Lambda - \Gamma}\left(x_\vee \hat{s}_{\Gamma'}, x_\vee (\hat{s}_{\Gamma'})_{\Lambda - \Gamma}\right), \quad (s,t) \in \Omega'_\Lambda \text{ and } j \notin \Lambda \tag{2.27}$$

Here μ^Γ is, as in Section 2.6, a finite signed measure on X_j. The notation $x_\vee \hat{s}_{\Gamma'}$ for $(x_\vee \hat{s})_{\Gamma'}$ should cause no confusion as $x \in X_j$ and $j \notin \Lambda$. The estimate (2.26) is then arrived at in this way: for each $\Gamma \neq \emptyset$ appearing in (2.27), set

$$\xi(s,t) = \tfrac{1}{2}\Big\{ \sup_x f^{\Lambda-\Gamma}\big[x_\vee \hat{s}_{\Gamma'}, (x_\vee \hat{s}_{\Gamma'})_{\Lambda-\Gamma}\big] + \inf_x f^{\Lambda-\Gamma}\big[x_\vee \hat{s}_{\Gamma'}, (x_\vee \hat{s}_{\Gamma'})_{\Lambda-\Gamma}\big]\Big\} \tag{2.28}$$

Then we have

$$\begin{aligned}\left|\int \mu^\Gamma(dx\,|\,s_{\Gamma'},\cdot)\, f^{\Lambda-\Gamma}(x_\vee \hat{s}_{\Gamma'},\cdot)\right| &= \left|\int \mu^\Gamma(dx\,|\,s_{\Gamma'},\cdot)\left\{f^{\Lambda-\Gamma}(x_\vee \hat{s}_{\Gamma'},\cdot) - \xi\right\}\right| \\ &\leqslant \|\mu^\Gamma(dx\,|\,s_{\Gamma'},\cdot)\|_{\text{tot. var}} \\ \tfrac{1}{2}\Big[\sup_x f^{f^{\Lambda-\Gamma}}(x_\vee \hat{s}_{\Gamma'},\cdot) - \inf_x f^{\Lambda-\Gamma}(x_\vee \hat{s}_{\Gamma},\cdot)\Big] &\leqslant R_{\Gamma,j}(\phi) D_{(\Lambda-\Gamma)\cup j}(f)\end{aligned} \tag{2.29}$$

where we have used the fact that μ^Γ has total mass zero, (2.23), and (2.12).

In case $\Gamma = \emptyset$, the corresponding term yields

$$\left|\int \mu(dx\,|\,s_{\Gamma'})\, f^{\Lambda}(x_\vee s_{\Gamma'},\cdot)\right| \leqslant \sup_x |f^\Lambda(x_\vee s_{\Gamma'},\cdot)| \leqslant D_\Lambda(f) \tag{2.30}$$

Finally, we note that there are $2^{|\Lambda|}$ different $\Gamma' \subset \Lambda$, each of which yields the same size estimate. When we take absolute values on both sides of (2.27) and a supremum over (s,t), the factor $2^{|\Lambda|}$ therefore cancels and we will have proved the theorem.

3. THE BOUNDEDNESS OF T_ϕ ON C^N; VASERSHTEIN MATRICES

The purpose of this section is to demonstrate that T_ϕ, as defined in Section 1, is a bounded linear operator on C^N, $N = 0, 1, 2, \ldots$, whenever the interaction ϕ is restricted to a suitable neighborhood $\mathscr{D}_N$ of the origin in P_{N+1}. In Section 5.3 we will show that $\phi \to T_\phi f : \mathscr{D}_N \to C^N$ is continuous. The proof of this rests on the arguments of Section 4. We first identify $\mathscr{D}_N$ and then state the main theorem.

3.1. The Dobrushin Uniqueness Regions

For N a natural number, let

$$\mathscr{D}_N = \big[\phi \in P_{N+1} : \alpha_1(\phi) < 1\big] \tag{3.1}$$

We call $\mathscr{D}_N$ the Nth Dobrushin uniqueness region. We will see in Section 4.2 that, on $\mathscr{P}_{N+1}$, $\alpha_M(\cdot)$, $1 \leqslant M \leqslant N$ are finite (and in fact continuous). In this section we shall assume this as an added condition in the definition of $\mathscr{D}_N$ above.

3.2. Theorem Bounding the $T_{\phi,k,p}$

For $\phi \in \mathscr{D}_N$, $N = 1, 2, \ldots$, the operators $T_{\phi,k,p}$ ($1 \leqslant k < \infty$, $k \leqslant p \leqslant \infty$), already defined in $C(\Omega)$, are bounded linear operators on C^N. For $f \in C^N$ and $\varphi \in \mathscr{D}_N$, we have (when $k < p$),

$$|T_{\varphi,k,p}f| \leqslant \rho_{k,p}(\varphi)|f| + \left[1 - \alpha_1(\varphi)\right]^{-1} \times \left(G \sum_{l=0}^{N-1} \left\{ G + \left[1 - \alpha_1(\varphi)\right]^{-1} JG \right\}^l \left[J|f| + E^p(f) \right] \right) \tag{3.2}$$

where $|f|$ is the $N \times 1$ column matrix $[|f|_M]$

$$\rho_{k,p}(\varphi) = \begin{cases} \alpha_1(\varphi), & \text{if} \quad k = 1 \quad \text{and} \quad p = \infty \\ 1, & \text{if} \quad k > 1 \quad \text{or} \quad p < \infty \end{cases} \tag{3.3}$$

where the $N \times N$ matrices G, J are defined by

$$G_{m,l} = \begin{cases} \alpha_{m-l+1}(\varphi), & 1 \leqslant l \leqslant m-1 \\ 0, & l \geqslant m \end{cases} \tag{3.4}$$

$$J_{m,l} = m\delta_{m,l} \qquad \text{(Kronecker } \delta\text{)} \tag{3.5}$$

and where $E^p(f)$ is the $N \times 1$ matrix whose Mth entry is

$$E^p_M(f) = \begin{cases} 0, & p = \infty \\ \left[1 - \alpha_1(\varphi)\right] \sum_{\substack{\Gamma \not\ni p \\ |\Gamma| = M-1}} D_{\Gamma \cup p}(f), & p < \infty \end{cases} \tag{3.6}$$

When $k = p$, we have

$$|T_{p,p}f|_N = |\zeta_p f|_N \leqslant |f|_N + \left[1 - \alpha_1(\varphi)\right]^{-1} \sum_{j=1}^{N} \alpha_{N-j+1} E^p_j(f) \tag{3.7}$$

3.3. Remarks on Theorem 3.2

(a) The estimate in (3.2) for $|T_{\varphi,k,p}f|_N$ involves the Nth seminorms of f only in the first term of the right-hand side, the matrix G being zero on and above its diagonal.

(b) The instance $N = 1$ was proved by L. Vasershtein.[5]

(c) We present estimate (3.2) for $N = 1$, 2, and 3. Set

$$\gamma_n(\phi) = \frac{\alpha_n(\phi)}{1 - \alpha_1(\phi)}, \qquad n = 1, 2, \ldots \tag{3.8}$$

Then, suppressing φ,

$$|T_{k,p}f|_1 \leqslant \rho_{k,p}|f|_1 \tag{3.9a}$$

$$|T_{k,p}f|_2 \leqslant \rho_{k,p}|f|_2 + \gamma_2[|f|_1 + E_1^p(f)] \tag{3.9b}$$

$$\begin{aligned} |T_{k,p}f|_3 \leqslant{}& \rho_{k,p}|f|_3 + 2\gamma_2|f|_2 + \left[\gamma_3 + \alpha_2\gamma_2\left(1 + \frac{2}{1-\alpha_1}\right)\right]|f|_1 \\ &+ \left\{\gamma_2 E_2^p + \left[\gamma_2\alpha_2\left(1 + \frac{2}{1-\alpha_1}\right) + \gamma_3\right]E_1^p\right\} \end{aligned} \tag{3.9c}$$

3.4. Definition of Vasershtein Matrices

Define

$$\begin{aligned} V_N &= \{\Lambda \subset L, |\Lambda| = N\}, \qquad N = 0,1,2,\ldots \\ U_N &= \bigcup_{n=1}^{N} V_n, \qquad N = 1,2,\ldots \end{aligned} \tag{3.10}$$

and

$$U_0 = \{\phi\} = V_0$$

For $\Lambda, \Gamma \subset L$, let

$$\chi_{\Lambda,\Gamma} = \begin{cases} 1 & \text{if } \Gamma \subset \Lambda \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

and let

$$\delta_{\Lambda,\Gamma} = \begin{cases} 1 & \text{if } \Gamma = \Lambda \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

In what follows we shall assume that ϕ is a fixed interaction (in $\mathscr{P}_1$, at least) and will not explicitly mention it in our formulas.

For fixed $N \geqslant 0$ and $\Lambda, \Gamma \in V_N$, put

$$\overset{(k)}{A}_{\Lambda,\Gamma} = \begin{cases} \delta_{\Lambda,\Gamma}, & \text{if } k \notin \Gamma \\ R_{\Lambda-(\Gamma-k),k}\chi_{\Lambda,\Gamma-k}, & \text{if } k \in \Gamma \end{cases} \tag{3.13}$$

and for $\Lambda \in V_N$, $\Gamma \in \bigcup_{N-1}$, put

$$\overset{(k)}{\mathscr{A}}_{\Lambda,\Gamma} = \begin{cases} 0, & \text{if } k \notin \Gamma \\ R_{A-(\Gamma-k),k}\chi_{\Lambda,r-k}, & \text{if } k \in \Gamma \end{cases} \tag{3.14}$$

[of course, $\overset{(k)}{\mathscr{A}}_{\Lambda,\Gamma}$ is defined by the right-hand side of (3.13) also]. Note that if $\Lambda \in V_1$, $\overset{(k)}{\mathscr{A}}_{\Lambda,\Gamma} \equiv 0$.

The $A^{(k)}, \mathscr{A}^{(k)}$ are here entitled "Vasershtein matrices" after L. Vasershtein, who studied them in the instance $N = 1$.[5]

3.5. Recasting Theorem 2.7

We may rewrite (2.26) in terms of Vasershtein matrices as follows: let $\Lambda \in V_N$. Then

$$D_\Lambda(\zeta_k f) \leqslant \sum_{\Gamma \in V_N} \overset{(k)}{A}_{\Lambda,\Gamma} D_\Gamma(f) + \sum_{\Gamma \in U_{N-1}} \overset{(k)}{\mathscr{A}}_{\Lambda,\Gamma} D_\Gamma(f), \qquad N \geqslant 1 \quad (3.15)$$

To see this, consider (2.26) (with $j \to k$). Note that

$$\sum_{\substack{\Gamma \subset \Lambda \\ \Gamma \neq \phi}} R_{\Gamma,k} D_{(\Lambda-\Gamma)\cup k} = \sum_{\substack{\Gamma' \subset \Lambda \\ \Gamma' \neq \Lambda}} R_{\Lambda-\Gamma',k} D_{\Gamma' \cup k} = \sum_{\substack{\Gamma : k \in \Gamma \\ \Gamma - k \subset \Lambda \\ |\Gamma| \leqslant N}} R_{\Lambda-(\Gamma-k),k} D_\Gamma$$

where we have successively set $\Gamma' = \Lambda - \Gamma$ and $\Gamma = \Gamma' \cup k$. Thus, employing (3.10) and (3.11), (2.26) translates to

$$D_\Lambda(\zeta_k f) \leqslant \begin{cases} 0, & k \in \Lambda \\ D_\Lambda(f) + \sum\limits_{\substack{\Gamma \in U_N \\ k \in \Gamma}} R_{\Lambda-(\Gamma-k),k} \chi_{\Lambda,\Gamma-k} D_\Gamma(f), & k \notin \Lambda \end{cases} \quad (3.16)$$

Now since U_N is the disjoint union of V_N and U_{N-1}, we see from the definitions (3.13) and (3.14) that (3.15) holds.

3.6. Products of the Matrices

Consider the following sum of nonnegative terms; for $\Lambda, \Gamma \in V_N$, $k = 1, 2, \ldots,$ and $n = 0, 1, 2, \ldots$:

$$B_{\Lambda,\Gamma}^{k,k+n} = \sum_{\Lambda_1 \in V_N} \cdots \sum_{\Lambda_n \in V_N} \overset{(k)}{A}_{\Lambda,\Lambda_1} \overset{(k+1)}{A}_{\Lambda_1,\Lambda_2} \cdots \overset{(k+n)}{A}_{\Lambda_n,\Gamma} \quad (3.17)$$

Doing the sum in the order $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$, we see that the sum is actually a finite one of at most $(N+1)^n$ terms ($\overset{(k)}{A}_{\Lambda,\Lambda_1}$, for example, stands a chance of being nonzero only if $\Lambda \not\ni k$ and either $\Lambda_1 = \Lambda$ or $\Lambda_1 = (\Lambda - j)_\cup k$ for some $j \in \Lambda$).

If we have $k + n \notin \Gamma$, then by definition $\overset{(k+n)}{A}_{\Lambda_n,\Gamma} = \delta_{\Lambda_n,\Gamma}$. Summing over the Λ_j from right to left (j decreasing) for those j with $k + j \notin \Gamma$, we have

$$B_{\Lambda,\Gamma}^{k,k+n} = \begin{cases} \delta_{\Lambda,\Gamma}, & \Gamma_\cap [k, k+n] = \emptyset \\ B_{\Lambda,\Gamma}^{k,\max\{\Gamma_\cap [k,k+n]\}}, & \Gamma_\cap [k, k+n] \neq \emptyset \end{cases} \quad (3.18)$$

where $[k, k+n] = \{k, k+1, \ldots, k+n\}$ and $\max \Lambda = \max\{j : j \in \Lambda\}$.

Now fix k, Λ, Γ and allow n to increase indefinitely. As soon as $n > \max \Gamma$, the sequence $B_{\Lambda,\Gamma}^{k,k+n}$ becomes constant and therefore has a limit

$B^{k,\infty}_{\Lambda,\Gamma}$:

$$B^{k,\infty}_{\Lambda,\Gamma} = \lim_{n\to\infty} B^{k,n}_{\Lambda,\Gamma} = \begin{cases} \delta_{\Lambda,\Gamma}, & \max\Gamma < k \\ B^{k,\max\Gamma}_{\Lambda,\Gamma}, & \max\Gamma \geqslant k \end{cases} \tag{3.19}$$

$$B^{k,\infty}_{\Lambda,\Gamma} = B^{k,j}_{\Lambda,\Gamma} \qquad \text{for any} \quad j \geqslant \max\Gamma \tag{3.20}$$

Certain summability estimates for these matrices were studied by Vasershtein and Gross[7] when $N = 1$. We will need similar estimates for $N \geqslant 1$. The next lemma enables us to reduce the estimations for $N > 1$ to those of Ref. 7.

3.7. Lemma

(a) Let Λ and Γ be in V_N for some $N \geqslant 1$ and let π_Λ, π_Γ be the sets of ordered arrangements of Λ, Γ, respectively. We write the components of $b \in \pi_\Lambda$ as b_i, $1 \leqslant i \leqslant N$. Then we have

$$B^{k,p}_{\Lambda,\Gamma} \leqslant \frac{1}{N!} \sum_{b\in\pi_\Lambda} \sum_{c\in\pi_\Gamma} \prod_{i=1}^{N} B^{k,p}_{b_i,c_i} = \sum_{b\in\pi_\Lambda} \prod_{i=1}^{N} B^{k,p}_{b_i,\mathscr{C}_i} = \sum_{c\in\pi_\Gamma} \prod_{i=1}^{N} B^{k,p}_{\mathscr{B}_i,c_i} \tag{3.21}$$

where $\mathscr{B}$ and $\mathscr{C}$ are any fixed elements of π_Λ, π_Γ, respectively.

(b) If $\max\Lambda \leqslant p < \max\Gamma$, $B^{k,p}_{\Lambda,\Gamma} = 0$.

The proof of part (a) rests on induction in p. Part (b) is a corollary to part (a) and the corresponding property for $N = 1$.

3.8. Lemma: Summability of the Matrices

(a) Let $N \geqslant 1$, $k \in [1,\infty)$ and $p \in [k,\infty]$. If $\alpha_1 < 1$, then for all $\Gamma \in V_N$,

$$\sum_{\Lambda\in V_N} B^{1,\infty}_{\Lambda,\Gamma} \leqslant \alpha_1 \tag{3.22}$$

$$\sum_{\Lambda\in V_N} B^{k,p}_{\Lambda,\Gamma} \leqslant 1 \tag{3.23}$$

and

$$\sum_{k=1}^{p-1} \sum_{\substack{\Lambda\in V_{N-1} \\ k\notin\Lambda}} B^{k+1,p}_{\Lambda\cup k,\Gamma} \leqslant \frac{N}{1-\alpha_1} \tag{3.24}$$

(b) Let $N \geqslant 2$ and $j \in [1, N-1]$. For each $k \in L$ and each $\Gamma \in V_{j-1}$ with $k \notin \Gamma$, we have

$$\sum_{\Lambda\in V_N} \overset{(k)}{\mathscr{A}}_{\Lambda,\Gamma\cup k} \leqslant \alpha_{N-j+1} \tag{3.25}$$

The proof of (3.22) is along the same lines as that in Gross [Ref. 7, Eq. (3.11)]. (3.23) follows by the same argument and (3.24) relies on Lemma 3.7 above and on equation (3.12) of Ref. 7 for its proof. Part (b) is a direct consequence of the definition (3.14) of $\mathscr{A}$.

The next lemma iterates the basic inequality (3.15) a finite number of times.

3.9. Lemma

For $k \in [1, \infty)$, $p \in [k, \infty)$, $N \geqslant 1$, $\Lambda \in V_N$ and $f \in C(\Omega)$, we have

$$D_\Lambda(T_{k,p}f) \leqslant \sum_{\Gamma \in V_N} B^{k,p}_{\Lambda,\Gamma} D_\Gamma(f) + \sum_{n=k-1}^{p-1} \sum_{\Gamma \in V_N} B^{k,n}_{\Lambda,\Gamma} \sum_{\substack{\Upsilon \not\ni (n+1) \\ \Upsilon \cup (n+1) \in U_{N-1}}} \overset{(n+1)}{A}_{\Gamma,\Upsilon \cup (n+1)} D_{\Upsilon \cup (n+1)}(T_{n+2,p}f) \tag{3.26}$$

where we make the conventions that $B^{k,k-1}_{\Lambda,\Gamma} = \delta_{\Lambda,\Gamma}$ and $T_{p+1,p}f = f$. In case $N = 1$, the second sum is considered empty and we have

$$D_i(T_{k,p}f) \leqslant \sum_{j=1}^{\infty} B^{k,p}_{i,j} D_j(f) \tag{3.27}$$

In order to extend this result to the case $p = \infty$, as well as to estimate $|T_{\varphi,k,p}f|_N$ as in (3.2), we need a technical lemma (for $N \geqslant 2$).

3.10. Lemma

Let $k \in [1, \infty)$, $p \in [k, \infty]$, n a natural number, $f \in C(\Omega)$ and $\phi \in \mathscr{D}_1$. Define

$$F_{k,p;n}(f) = (1 - \alpha_1) \sum_{m=k}^{p} \sum_{\substack{\Lambda \in V_{n-1} \\ m \notin \Lambda}} D_{\Lambda \cup m}(T_{m+1,p}f) \tag{3.28}$$

where, as usual, $T_{p+1,p}f = f$ $(p < \infty)$.

Then for $p < \infty$ we have

$$F_{k,p;n}(f) \leqslant \begin{cases} n|f|_n + \left(1 + \dfrac{n}{1-\alpha_1}\right) \displaystyle\sum_{l=1}^{n-1} \alpha_{n-l+1} F_{k+1,p;l}(f) & \\ \qquad + E^p_n(f), & \text{if } p > k \\ E^p_n(f) & \text{if } p = k \end{cases} \tag{3.29}$$

[where E^p_n is defined as in (3.6)]. The sum above is zero when $n = 1$.

We may express the estimate for F in matrix form. To this end, let F, $|f|$, and E^p be $n \times 1$ column matrices whose mth entries are $F_{k,p;m}$, $|f|_m$, and E^p_m respectively. Let G and J be the $n \times n$ matrices defined in (3.4) and (3.5) with $N = n$. Then if $k < p < \infty$ we have for $j \in [1, n]$

$$F_j \leqslant \sum_{k=0}^{j-1} \left\{ \left[\left(1 + \frac{1}{1-\alpha_1} J \right) G \right]^k (J|f| + E^p) \right\}_j \tag{3.30}$$

The proof of this lemma involves a careful application of Lemmas 3.8 and 3.9 to the summand in (3.28). We then get (3.30) by noting that $F_{k+1,p,l}(f) \leqslant F_{k,p,l}(f)$ and that if two matrices are both zero on and above their diagonals, then so is their product.

3.11. Proof of Theorem 3.2

Our task is to show that the conclusions of Lemma 3.9 and 3.10 continue to hold when $p = \infty$ and then to sum (3.26) over $\Lambda \in V_M$, $1 \leqslant M \leqslant N$. To this end, let

$$R_\Lambda^{k,p}(f) = \text{right-hand side of (3.26)} \tag{3.31}$$

Since this is a sum of nonnegative (though possibly infinite) terms, it is well defined for $p = \infty$. We will first show that

$$\begin{aligned} D_\Lambda(T_k f) &= \lim_{p\to\infty} D_\Lambda(T_{k,p} f) \leqslant \lim_{p\to\infty} R_\Lambda^{k,p}(f) \\ &= R_\Lambda^{k,\infty}(f) < \infty \end{aligned} \tag{3.32}$$

The first equality follows from the fact that $D_\Lambda(\cdot)$ is a continuous seminorm on $C(\Omega)$ [cf. Remark 2.4(a)]. Thus we are required to show that, under our hypotheses $\phi \in D_N$, $f \in C^N$,

$$\lim_{p\to\infty} R_\Lambda^{k,p}(f) = R_\Lambda^{k,\infty}(f) < \infty \tag{3.33}$$

We do this by induction on N as follows:

(i) We show that (3.33) holds for $N = 1$ and then make the induction hypothesis that it holds for $N \leqslant M - 1$.

(ii) It will follow that Lemma 3.10 holds for $p = \infty$, $n \leqslant m - 1$.

(iii) This will yield a bound for $F_{k,p;M-1}$, $p \leqslant \infty$, with which we will establish (3.33) when $|\Lambda| = M$.

(i) Let $N = 1$. Then by (3.27), if $p < \infty$ and $i \in L$,

$$R_i^{k,p}(f) = \sum_{j=1}^{\infty} B_{i,j}^{k,p} D_j(f) \tag{3.34}$$

Now suppose $p > i$. By Lemma 3.7(b) and (3.20),

$$R_i^{k,p}(f) = \sum_{j=1}^{p} B_{i,j}^{k,p} D_j(f) = \sum_{j=1}^{p} B_{i,j}^{k,\infty} D_j(f), \qquad (i < p < \infty) \quad (3.35)$$

Hence

$$\lim_{p\to\infty} R_i^{k,p}(f) = \sum_{j=1}^{\infty} B_{i,j}^{k,\infty} D_j(f) = R_k^{k,\infty}(f) \quad (3.36)$$

Finally, Lemma 3.8 shows us that

$$R_i^{k,\infty}(f) \leqslant \sum_j R_j^{k,\infty}(f) \leqslant \rho_{k,\infty}(\phi)|f|_1 < \infty \quad (3.37)$$

where $\rho_{k,\infty}(\phi)$ is defined in (3.3).

Thus (3.33) holds for $N = 1$. Assume that, for some $M \geqslant 1$, it holds for all $N \leqslant M - 1$, $\Lambda \in V_N$ ($f \in C^N$ and $\phi \in \mathscr{D}_N$).

Let now f be in C^M and $\phi \in \mathscr{D}_M$. Then *a priori* $f \in C^N$, $\phi \in \mathscr{D}_N$ for each $N \leqslant M$.

(ii) By the above induction hypothesis, (3.26) holds with $p = \infty$ and $|\Lambda| = j \leqslant M - 1$. An examination of the proof of Lemma 3.10 reveals that, in that case, the statement of Lemma 3.10 continues to be true with $p = \infty$ (and $E^\infty = 0$) for $n \leqslant M - 1$. Thus, we have that, for $1 \leqslant n \leqslant M - 1$,

$$F_{k,\infty;n}(f) \leqslant \sum_{l=1}^{n} K_{n,l}|f|_l \quad (3.38)$$

with

$$K_{n,l} = \sum_{m=0}^{M-1} \left\{ \left[\left(I + \frac{1}{1-\alpha_1} J \right) G \right]^m J \right\}_{n,l} \quad (3.39)$$

where J, G are as in (3.5). It is permissible to extend this sum over m to $M - 1$ (rather than $n - 1$) since the summands are zero for $m \geqslant n - 1$. Since $\phi \in \mathscr{D}_M$, all entries are finite and

$$F_{k,p;n}(f) < \infty \qquad \text{for} \quad f \in C^M, \quad \phi \in \mathscr{D}_M, \quad n \leqslant M - 1, \quad \text{and} \quad p \leqslant \infty \quad (3.40)$$

(iii) Fix $\Lambda \in V_M$ and put, for any $h \in C(\Omega)$,

$$P_\Lambda^{k,p}(h) = \sum_{n=k}^{p} \sum_{\Gamma \in V_M} B_{\Lambda,\Gamma}^{k,n-1} \sum_{\substack{\Upsilon \not\ni n \\ \Upsilon \cup n \in U_{M-1}}} \overset{(n)}{\mathscr{A}}_{\Gamma,\Upsilon\cup n} D_{\Upsilon\cup n}(T_{n+1,p} h) \quad (3.41)$$

i.e., the second term of $R_\Lambda^{k,p}(h)$ (we have adjusted the index "n" a bit). As usual, $B_{\Lambda,\Gamma}^{k,k-1} = \delta_{\Lambda,\Gamma}$ and $T_{p+1,p} f = f$ for $p < \infty$.

Now suppose that $p > \max\Lambda$. Then by Lemma 3.7(b) and (3.20) we have

$$R_\Lambda^{k,p}(h) = \sum_{\substack{\Gamma\in V_M\\ \max\Gamma\leqslant p}} B_{\Lambda,\Gamma}^{k,\infty} D_\Gamma(h) + P_\Lambda^{k,p}(h) \tag{3.42}$$

Thus, to prove (3.33) it suffices to show that

$$\lim_{p\to\infty} P_\Lambda^{k,p}(f) = P_\Lambda^{k,\infty}(f) < \infty \qquad (|\Lambda| = M) \tag{3.43}$$

Consider $P_\Lambda^{k,p}(h)$. From (3.41), (3.23), and (3.25) we have, for $k \leqslant p \leqslant \infty$, that

$$\begin{aligned} P_\Lambda^{k,p}(h) &\leqslant \sum_{\Lambda'\in V_M} P_{\Lambda'}^{k,p}(h) \leqslant \sum_{j=1}^{M-1}\sum_{n=k}^{p} \alpha_{M-j+1} \sum_{\substack{\Upsilon\\ \Upsilon\cup n\in V_j}} D_{\Upsilon\cup n}(T_{n+1,p}h) \\ &\leqslant \frac{1}{1-\alpha_1}\sum_{j=1}^{M-1} \alpha_{M-j+1} F_{k,p;j}(h) < \infty \quad \text{if } h\in C^M \text{ and } k\leqslant p\leqslant\infty \end{aligned} \tag{3.44}$$

as follows from (3.40) above.

Suppose now that g is a cylinder function based on q. Since $P_\Lambda^{k,p}(h)$ is finite for $h \in C^M$, $P_\Lambda^{k,p}(\cdot)$ is a seminorm for each $p < \infty$, so that if $q < p < \infty$,

$$|P_\Lambda^{k,p}(f) - P_\Lambda^{k,\infty}(f)| \leqslant P_\Lambda^{k,p}(f-g) + P_\Lambda^{k,\infty}(f-g) \tag{3.45}$$

Here we have used $P_\Lambda^{k,p}(g) = P_\Lambda^{k,\infty}(g)$ for $p > q$.

Now it follows from (3.44) and (3.38) (and its analog with $p < \infty$) that $P^{k,p}(\cdot): C^M \to R^1$ is continuous at 0 for each $p \leqslant \infty$, under our hypothesis that $\phi \in \mathscr{D}_M$. Moreover, the bounds on $P_\Lambda^{k,p}(h)$ may be chosen uniformly in p, since $E_j^p(h) \to 0$ as $p \to \infty$ for $j \leqslant M$, $h \in C^M$. Hence we may make the left-hand side of (3.45) as small as desired by first choosing a cylinder function g close to f (in the $|\cdot|_n$ seminorms, $1 \leqslant n \leqslant M$) and then letting p go to ∞. We have therefore established (3.43) for $\Lambda \in V_M$ and therefore also (3.33).

It remains to estimate $|T_{k,p}f|_N$. As in (3.44), we have from (3.26) (extended to $p \leqslant \infty$) that

$$\begin{aligned} |T_{k,p}f|_N &= \sum_{\Lambda\in V_N} D_\Lambda(T_{k,p}f) \\ &\leqslant \sum_\Gamma\sum_\Lambda B_{\Lambda,\Gamma}^{k,p} D_\Gamma(f) + \frac{1}{1-\alpha_1}\sum_{j=1}^{N-1} \alpha_{N-j+1} F_{k,p;j}(f) \\ &\leqslant \rho_{k,p}|f|_N + \frac{1}{1-\alpha_1}(GF)_N \end{aligned} \tag{3.46}$$

for $1 \leqslant k \leqslant p \leqslant \infty$. We have used (3.22), (3.23), and (3.28). Now applying (3.30) of Lemma 3.10 (and letting the sum over k extend to $N-1$ there [see remarks after (3.39)], we conclude that (3.2) holds and its right-hand side is finite. As for (3.7), simply estimate (3.15) as above [see Theorem 4.4(a)(ii)].

3.12. Remarks

It follows, as in the proof above, that in fact

$$\lim_{p\to\infty} |T_{k,p}f|_N = |T_k f|_N, \qquad f \in C^N, \quad \phi \in \mathscr{D}_N \tag{3.47}$$

since by the estimates of Theorem 3.2, $|T_{k,p} \cdot|$ is a seminorm on C^N for each $p \leqslant \infty$, when $\phi \in D_N$.

The $N=1$ estimate

$$|T_\phi f|_1 \leqslant \alpha_1(\phi)|f|_1, \qquad \phi \in \mathscr{D}_1 \tag{3.48}$$

may be used to prove Dobrushin's theorem and Vasershtein's formula (1.5). See Ref. 7, Corollary 3.3 or Ref. 23, pp. 61 and 62.

4. DERIVATIVES OF $\phi \to \zeta_j^\phi f$

4.1. Existence of Derivatives of $\phi \to \zeta_j^\phi f : \mathscr{P}_1 \to C(\Omega)$; Ursell Functions

In the following, all interactions will be in $\mathscr{P}_1$ and j is a fixed point of L. Given an interaction ψ, write

$$k_\psi(\cdot) = -\sum_{\Lambda \in j} \psi(\cdot \mid \Lambda) \tag{4.1}$$

As in Section 2.5, $k_\psi \in C(\Omega)$.

Let $g = k_\phi$, $h \in C(\Omega)$ and $s \in \Omega_{L-j}$. Put

$$Z_g(s) = \int_{X_j} \nu_j(dx)\, e^{g(x \vee s)}$$

and

$$\langle h \rangle_g(s) = Z_g^{-1}(s) \int_{X_j} \nu_j(dx)\, e^{g(x \vee s)} h(x \vee s) \big(\equiv \zeta_j^\phi h(s)\big) \tag{4.2}$$

Now suppose the function g in (4.2) depends also on some parameter $v \in I$, $I \subset R^1$ an open interval. We claim that if $v \to g_v : I \to C(\Omega)$ is differentiable at $v_0 \in I$, then so is $v \to \langle h \rangle_{g_v} : I \to C(\Omega)$, with derivatives given by

$$\frac{d}{dv} \langle h \rangle_{g_v} \big|_{v=v_0} = \langle h\big(g'_{v_0} - \langle g'_{v_0} \rangle_{g_{v_0}}\big) \rangle_{g_{v_0}} \tag{4.3}$$

where $g'_{v_0} = (dg_v/dv)|_{v=v_0}$. This may be seen by means of the usual dominated convergence arguments. More generally we may suppose that v lies in some open ball of R^n; the partial derivatives of $v \to \langle h \rangle_{g_v}$ are then found by successive differentiations of the type (4.3). Finally, with suitable modifications to (4.3), we may allow h to depend upon v also.

Our concern is with Gateaux derivatives of $\phi \to \zeta_j^\phi f$, so that we now specialize to functions of the form

$$v \to g_{\mathbf{v}} = g + \mathbf{v} \cdot \mathbf{k} = g + \sum_{i=1}^{n} v_i k_i \tag{4.4}$$

where $\mathbf{v} = (v_1, \ldots, v_n) \in R^n$ and $\mathbf{k} = (k_1, \ldots, k_n) \in (C(\Omega))^n$. (The k_i will arise as k_{ψ_i} [see (4.1)] or as A_{ψ_i} (2.18).) Note that $g_{\mathbf{v}} \in C(\Omega)$ for $\mathbf{v} \in R^n$ and by the above remarks, $\mathbf{v} \to \langle h \rangle_{g_{\mathbf{v}}}$ is C^∞ differentiable.

We introduce the notation

$$\left(\prod_{i=1}^{m} \partial_i\right)\langle h \rangle_g = \left(\frac{\partial^m}{\partial v_1 \cdots \partial v_m} \langle h \rangle_{g_{\mathbf{v}}}\right)\Big|_{\mathbf{v}=0} \tag{4.5}$$

when $1 \leqslant m \leqslant n$. It is clear that the order of differentiation is immaterial and that

$$\partial_1 \langle h \rangle_g = \langle h k_1 \rangle_g - \langle h \rangle_g \langle k_1 \rangle_g \tag{4.6}$$

We may characterize the derivatives (4.5) as follows.

For $f_i \in C(\Omega)$, $1 \leqslant i \leqslant m$, define the mth-*order Ursell function* [with respect to $g \in C(\Omega), j \in L$] by

$$\begin{aligned} u_m^g(f_1, \ldots, f_m)(s) &= u_m^g(\{f_i\}_1^m)(s) \\ &= \sum_{Q_m} (-1)^{|Q_m|-1}(|Q_m| - 1)! \prod_{P \in Q_m} \langle f_P \rangle_g(s) \\ &\qquad (s \in \Omega_{L-j}), \end{aligned} \tag{4.7}$$

where Q_m runs over all partitions of the set $[1, m]$ and $f_P = \prod_{i \in P} f_i$ (see, e.g., Ref. 24). The significance of Ursell functions is that

$$\prod_{i=1}^{n} \partial_i \langle h \rangle_g = u_{n+1}^g(h, k_1, \ldots, k_n) \tag{4.8}$$

This may be verified by induction on n, using (4.6) [see (4.10) below]. Note that $u_m^g(\{f_i\})(\cdot) \in C(\Omega)$.

For later purposes we need to generalize somewhat the notion of Ursell functions. Note from (4.2) that the right-hand side of (4.1) is still well defined when $g \in C(X_j \times M), f_i \in C(X_j \times M_i)$, $1 \leqslant i \leqslant n$, where M, M_i are topological spaces (of course, the argument "s" has to be suitably altered). In particular, our M_i will be of the form $\Omega'_\Lambda \times (\times_{i=1}^{n} \Omega'_{\Lambda_i})$ or some suitable subset.)

We mention some properties of Ursell functions.

Remarks 4.1.1. (a) If $m \geqslant 2$ and some f_l is independent of the jth coordinate, $u_m^g(\{f_i\}) \equiv 0$. This follows from the representation (4.8): if $k_1(x_\vee s)$ is independent of x, $\partial_1\langle h\rangle_g \equiv 0$ for all g.

It is clear that $\{f_i\}_{i=1}^n \to u_n^g(\{f_i\}) : (C(\Omega))^n \to C(\Omega)$ is multilinear. This and the above remark allow us to, for example, replace any f_i in the argument by $f_i - \langle f_i\rangle_g$ without affecting the value of u_m^g.

(b) Letting Q_{m-1} denote partitions of $[1, m-1]$ and putting $C_{Q_m} = (-1)^{|Q_m|-1}(|Q_m|-1)!$, we have

$$u_m^g(\{f_i\}_i^m) = \sum_{Q_{m-1}} C_{Q_{m-1}} \sum_{P \in Q_{m-1}} \langle f_P(f_m - \langle f_m\rangle)\rangle_g \prod_{\substack{S \in Q_{m-1} \\ S \neq P}} \langle f_S\rangle_g \tag{4.9}$$

This may be seen by differentiation of $v \to u_{m-1}^{g+vf_m}(\{f_i\}_1^{m-1})$, or combinatorically by generating the Q_m from the Q_{m-1}, by adding the element m in $|Q_{m-1}|+1$ ways.

The remarks in (a) also follow from (4.9).

(c) Moving back to the general situation in which

$$v \to g_v, \qquad v \to f_{i,v}, \qquad 1 \leqslant i \leqslant m$$

are differentiable from an open set $I \subset R^1$ into $C(\Omega)$, we have that

$$\frac{d}{dv} u_m^{g_v}(\{f_{i,v}\}_1^m) = u_{m+1}^{g_v}\left(\frac{dg_v}{dv}, \{f_{i,v}\}\right) + \sum_{i=1}^m u_m^{g_v}\left(\frac{df_{i,v}}{dv}, \{f_{j,v}\}_{j \neq i}\right) \tag{4.10}$$

This is an expression of Leibnitz' product rule on the symbols $u_m^{g_v}(\ )$, $f_{1,v}, \ldots, f_{m,v}$.

4.2. Relative Hamiltonian Spaces

We will find it most convenient here to work with the so-called relative Hamiltonian norms on our interactions rather than the usual $\mathscr{P}_n$ norms. Let

$$\Omega_F = \bigcup\{\Omega'_\Lambda, \Lambda \text{ finite}\} \tag{4.11}$$

Then we define $\mathscr{H}_1$ to be the space of all continuous real-valued functions $H(\cdot,\cdot)$ on Ω_F satisfying

(i) $H(s,t) = -H(t,s)$

(ii) $H(s,t) = H(s,u) + H(u,t)$ whenever $(s,t) \in \Omega'_\Lambda$ and $(s,u),(u,t) \in \Omega'_\Lambda$, for some $\Lambda \subset L$

(iii) $$|H|_1 = \sup_{a \in L}(\sup\{|H(s,t)| : s = t \text{ off } a\}) < \infty \tag{4.12}$$

$\mathscr{H}_1$ is the first relative Hamiltonian space, so named because an interaction $\phi \in \mathscr{P}_1$ gives rise to an element of $\mathscr{H}_1$ by

$$H_\phi(s,t) = \sum_{\Lambda \subset L} \left[\phi(s \mid \Lambda) - \phi(t \mid \Lambda)\right] \tag{4.13}$$

Note that $\mathscr{H}_1$ equipped with the norm $|\cdot|_1$ is a Banach space and that the canonical map (4.13) takes $\mathscr{P}_1$ into $\mathscr{H}_1$ continuously—indeed $|H_\varphi|_1 \leqslant 2\|\varphi\|_1$. As is to be expected, relative Hamiltonians are more closely related to their Gibbs states than are interactions[26,27].

Next, define for $H \in \mathscr{H}_1, j \in L$ and $\Lambda \subset L - j$

$$\lambda^H(j,\Lambda) = \sup\left[\left|\sum_{\Gamma \subset \Lambda \cup j} (-1)^{|\Gamma|} H(s_\Gamma, t)\right| : (s,t) \in \Omega'_{\Lambda \cup j}\right] \tag{4.14}$$

For instance, if $\Lambda = \{i\} \neq \{j\}$, $s = s_i \vee s_j \vee \hat{s}$ and $t = t_i \vee t_j \vee \hat{s}$,

$$\lambda^H(j,\{i\}) = \sup\{|H(s_i \vee s_j \vee \hat{s}, t) - H(t_i \vee s_j \vee \hat{s}, t) - H(s_i \vee t_j \vee \hat{s}, t)| : (s,t) \in \Omega'_{\{i,j\}}\} \tag{4.15}$$

If $\Lambda = \varnothing$, $\lambda^H(j,\varnothing) = \sup\{|H(s,t)| : s = t \text{ off } j\} \leqslant |H|_1$. Define the nth relative Hamiltonian norm by

$$|H|_n = \sup_j \sum_{\substack{\Lambda \subset L-j \\ |\Lambda| = n-1}} \lambda^H(j,\Lambda), \qquad n = 1,2,\ldots \tag{4.16}$$

and the Banach space $\mathscr{H}_n$ by

$$\mathscr{H}_n = \{H \in \mathscr{H}_1 : |H|_m < \infty, 1 \leqslant m \leqslant n\} \tag{4.17}$$

with norm, say,

$$\|H\|_n = \sum_{m=1}^{n} |H|_n$$

If $H = H_\phi$ we will write λ^ϕ for λ^{H_ϕ}. $\lambda^\phi(i,j)$ measures the rate of decay of the potential: putting (4.13) and (4.14) together one sees that

$$\lambda^\phi(j,\Lambda) \leqslant 2^{|\Lambda|+1} \sup_s \left|\sum_{A \supset j \cup \Lambda} \phi(s \mid \Lambda)\right| \tag{4.18}$$

The right-hand side measures the strength of interaction between "spins" in $j \cup \Lambda$. If, for example, ϕ has finite range R, then $\lambda^\phi(j,\Lambda) = 0$ whenever the distance between j and Λ is greater than R. From (4.18) we may derive

$$|H_\phi|_n = \sup_j \sum_{\substack{\Lambda : j \notin \Lambda \\ |\Lambda| = n-1}} \lambda^\phi(j,\Lambda) \leqslant 2^n \sup_j \sum_{A : j \in A} \binom{|A|-1}{n-1} |\phi(\cdot \mid A)|_\infty \tag{4.19}$$

so that the canonical map (4.13) takes P_n to $\mathscr{H}_n$ continuously.

The estimates we shall come across in the sequel involve expressions of the form $D_{j\cup\Lambda}(k_\psi)$, k_ψ as in (4.1). The relative Hamiltonian norms are particularly well suited to these since (we are now fixing j as in Section 4.1)

$$D_{j\cup\Lambda}(k_\psi) = \lambda^{\psi}(j,\Lambda), \qquad j \notin \Lambda \tag{4.20}$$

and in particular

$$D_j(k_\psi) \leqslant |H_\psi|_1 \tag{4.21}$$

To see this, note that

$$D_{j\cup\Lambda}(k_\psi) = \sup_{(s,t)\in\Omega'_{\Lambda\cup j}} \left| \sum_{A\,:\,j\in A} \psi^{j\cup\Lambda}(s,t\mid A)\right| = \sup\left|\sum_{A\subset L}\psi^{j\cup\Lambda}(s,t\mid A)\right| \tag{4.22}$$

by Property 2.3(e). But

$$\begin{aligned}\sum_{A\subset L}\psi^{j\cup\Lambda}(s,t\mid A) &= \sum_{A\subset L}\sum_{\Gamma\subset j\cup\Lambda}(-1)^{|\Gamma|}\psi(s_\Gamma\mid A)\\ &= \sum_{A\subset L}\sum_{\Gamma\subset j\cup A}(-1)^{|\Gamma|}[\psi(s_\Gamma\mid A)-\psi(t\mid A)]\\ &= \sum_{\Gamma\subset j\cup\Lambda}(-1)^{|\Gamma|}H_\psi(s_\Gamma,t)\end{aligned} \tag{4.23}$$

since $\sum_\Gamma(-1)^{|\Gamma|}=0$. Finally, use (4.14).

We are now in a position to state the two main results of this section.

4.3. Proposition on $\alpha_N(\phi)$

For $N=1,2,\ldots,$ α_N, as defined in Section 2.6, is a continuous real-valued function on $\mathscr{P}_{N+1}$. (Hence $\mathscr{D}_N$ as defined in 3.1 is an open neighborhood of the origin in P_{N+1}.)

The α_N are in fact "locally Lipschitz": e.g., for $N=1,2$ we have for $\phi,\phi'\in P_{N+1}$, $H=H_\phi$ and $H'=H_{\phi'}$,

$$|\alpha_1(\phi)-\alpha_1(\phi')| \leqslant \tfrac{1}{2}|H-H'|_2+\tfrac{1}{2}|H-H'|_1(|H|_2+|H'|_2) \tag{4.24a}$$

$$\begin{aligned}|\alpha_2(\phi)-\alpha_2(\phi')| \leqslant{}& |H-H'|_3+\tfrac{1}{2}|H-H'|_2(|H|_2+|H'|_2)\\ &+|H-H'|_1\big(\tfrac{1}{2}|H|_3+\tfrac{4}{3}|H|_2^2+\tfrac{4}{3}|H|_2|H'|_2\\ &+\tfrac{4}{3}|H'|_2^2+\tfrac{1}{2}|H'|_3\big)\end{aligned} \tag{4.24b}$$

4.4. Theorem (Continuity of $\phi\to\zeta_j^\phi f$ and Its Derivatives)

Fix $j\in L$

(a) Let $N=0,1,2,\ldots$. If $f\in C^N$ and $\phi\in\mathscr{P}_{N+1}$,

(i) $\zeta_j^\phi f\in C^N$ $(C^0=C(\Omega))$.

(ii) $$|\zeta_j^\phi f|_N \leqslant |f|_N + \sum_{m=1}^{N} \alpha_m(\phi) \sum_{\Gamma \in V_{N-m}, j \notin \Gamma} D_{\Gamma \in j}(f) \tag{4.25}$$

(where the sum is zero for $N = 0$).

(iii) $\phi \to \zeta_j^\phi f : P_{N+1} \to C^N$ is continuous.

(b) Let n be a natural number, $\psi_i \in \mathscr{P}_1$ for $1 \leqslant i \leqslant n$, $\phi \in \mathscr{P}_1$ and $f \in C(\Omega)$. Then

(i) The function $(u_1, \ldots, u_n) \to \zeta_j^{\phi + \Sigma u_i \psi_i} f : R^n \to C(\Omega)$ possesses the nth derivative

$$\begin{aligned} \partial_{\psi_1, \ldots, \psi_n} \zeta_j^\phi f &= \frac{\partial^n}{\partial u_1 \ldots \partial u_n} \zeta_j^{\phi + \Sigma u_i \psi_i} f \Big|_{u_1 = \cdots u_n = 0} \\ &= u_{n+1}^g (f, k_1, \ldots, k_n) \end{aligned} \tag{4.26}$$

where $g = k_\phi$, $k_1 = k_{\psi_i}$ as in (4.1). Moreover, $\phi \to \partial_{\psi_1, \ldots, \psi_n} \zeta_j^\phi f : \mathscr{P}_1 \to C(\Omega)$ is continuous for fixed $f \in C(\Omega)$.

(ii) If in addition $\phi, \psi_1 \in \mathscr{P}_{N+1}$ $(N = 0, 1, \ldots)$ and $f \in C^N$, $\partial_{\psi_1, \ldots, \psi_n} \zeta_j^\phi f$ is in C^N. Moreover, $\phi \to \partial_{\psi_1, \ldots, \phi_n} \zeta_j^\phi f$ is continuous from $\mathscr{P}_{N+1}$ into C^N, for fixed $f \in C^N$, $\psi_i \in \mathscr{P}_{N+1}$.

Estimates of the derivatives are as follows: for $M = 0, 1, \ldots,$

$$|\partial_{\psi_1, \ldots, \psi_n} \zeta_j^\phi f|_M \leqslant \sum_{m=1}^{M+1} C_{M,m}(\phi, \{\psi_i\}) \sum_{\substack{\Lambda \in V_{m-1} \\ j \notin \Lambda}} D_{j \cup \Lambda}(f) \tag{4.27n}$$

where each $C_{M,m}(\phi, \{\psi_i\})$ is a polynomial in $|H_\phi|_l, |H_{\psi_i}|_l$ for $1 \leqslant l \leqslant M + 1$ (and $1 \leqslant i \leqslant n$).

Note that the above estimate is summable over j if $f \in C^{M+1}$.

In concluding the statement of the lemma, we give estimates of those derivatives of $\zeta_j^\phi f$ needed for establishing the second derivative of the Gibbs state. Let $H^1 = |H_{\psi_1}|$ etc. Then

$$|\partial_\psi \zeta_j^\phi f|_\infty \leqslant D_j(f) \min(\|\psi\|_1, |H_\psi|_1) \tag{4.28}$$

$$|\partial_{\psi_1, \psi_2} \zeta_j^\phi f|_\infty \leqslant D_j(f) |H^1|_1 |H^2|_1 \tag{4.29}$$

$$|\partial_\psi \zeta_j^\phi f|_1 \leqslant D_j(f)(|H_\psi|_2 + |H_\psi|_1 |H_\psi|_2) + \sum_{\substack{l \\ l \neq j}} D_{j,l}(f) |H_\psi|_1 \tag{4.30}$$

4.5. Estimating $|u_n(\{f_i\})|$

As we have seen in Section 4.1, we may write

$$u_n^g(\{f_i\}_1^n) = \sum_{Q_n}{}' (-1)^{|Q_n|} (|Q_n| - 1)! \prod_{S \in Q_n} \left\langle \prod_{i \in S} (f_i - \langle f_i \rangle_g) \right\rangle_g \tag{4.31}$$

where the prime denotes a sum over partitions without a singleton. Since $\sup_{x\in X_j}|f_i(x_\vee s) - \langle f_i\rangle_g(s)| \leqslant D_j(f_i)$, we see that

$$|u_n^g(\{f_i\}_1^n)| \leqslant \Big[\sum_{Q_n}{}'(|Q_n|-1)!\Big]\prod_{i=1}^n D_j(f_i) = t_n\prod_{i=1}^n D_j(f_i) \qquad (4.32)$$

By means of the inclusion–exclusion principle one can derive a formula for t_n (Ref. 23, Section 4.11), namely, that $t_1 = 0$, $t_2 = 1$ and if $n \geqslant 3$,

$$t_n = \sum_{Q_n}{}'(|Q_n| - 1!)$$

$$= \sum_{r=1}^{[n/2]}\sum_{k=0}^{r-1}\sum_{l=k}^{r-1} k!\binom{n}{k}\binom{r-1}{l}\binom{l}{k}(r-l)^{n-k-1}(-1)^l \qquad (4.33)$$

(4.32) is the fundamental estimate of this section: we will see that all higher-order differences may be expressed in terms of it.

We note here that, in the case $n = 2$, a better estimate may be obtained by means of Schwartz' inequality (as in Ref. 3, Lemma 2), namely,

$$|u_2^g(f_1, f_2)| \leqslant \min\big[|f_1|_\infty, D_j(f_1)\big]\cdot D_j(f_2) \qquad (4.34)$$

We remark that the estimates (4.32) are themselves a consequence of

$$|u_n| \leqslant t_n\prod_{i=1}^n \sup_{x,x'\in X_j}|f_i^{[j]}(x_v s, x'_v s)| \qquad (4.35)$$

with $s\in\Omega_{L-j}$. Also, the estimates, suitably modified, continue to hold for the situation (mentioned in Section 4.1) in which, for example, $f_i \in C(X\times M_i)$.

4.6. A Representation for Higher-Order Differences of Ursell Functions

We consider the functions $u_{m+1}(\cdot)$, corresponding to the mth derivative, $m \geqslant 1$. Let $\Lambda\subset L$, $|\Lambda|\geqslant 1$. We may suppose that $j\notin\Lambda$, since otherwise $[u_{m+1}^g(\cdot)]^\Lambda \equiv 0$. Let $(s,t)\in\Omega'_\Lambda$ and with respect to this pair, put

$$\begin{cases} g_\Gamma(x) = g(x_\vee \hat{s}_\Gamma) \\ k_{i,\Gamma}(x) = k_i(x_\vee \hat{s}_\Gamma), \qquad \Gamma\subset\Lambda, \quad 1\leqslant i\leqslant m+1, \quad x\in X_j \end{cases} \qquad (4.36)$$

Let $v = \{v_a\}_{a\in\Lambda}$, $0\leqslant v_a\leqslant 1$ for each $a\in\Lambda$. Put

$$K(\Gamma,\Gamma') = \prod_{a\in\Gamma'}(1-v_a)\prod_{b\in\Gamma-\Gamma'} v_b, \qquad \Gamma'\subset\Gamma\subset\Lambda \qquad (4.37)$$

where the empty product is assumed to be unity. Finally, put

$$g_v = \sum_{\Gamma \subset \Lambda} K(\Lambda, \Gamma) g_\Gamma \tag{4.38}$$

and similarly for the k_i.

We note the following properties;

(i) $$\sum_{\Gamma' \subset \Gamma} K(\Gamma, \Gamma') = 1 \qquad \text{for all} \quad \Gamma \subset \Lambda \tag{4.39}$$

(ii) $$K(\Gamma, \Gamma') = \begin{cases} (1 - v_c) K(\Gamma - c, \Gamma' - c), & \text{if} \quad c \in \Gamma' \\ v_c K(\Gamma - c, \Gamma'), & \text{if} \quad c \in \Gamma - \Gamma' \end{cases} \tag{4.40}$$

(iii) If v is such that $v_a = 0$ for all $a \in \Gamma'$ and $v_b = 1$ for all $b \in \Gamma - \Gamma'$, then

$$K(\Lambda, \Gamma) = \begin{cases} 1 & \text{if} \quad \Gamma = \Gamma' \\ 0 & \text{otherwise} \end{cases} \tag{4.41}$$

Hence

(iv) $$g_v = g_\Gamma \qquad \text{if} \quad v_a = \begin{cases} 0, & a \in \Gamma \\ 1 & \text{otherwise} \end{cases} \tag{4.42}$$

Of these, (ii)–(iv) are straightforward consequences of the definitions (4.37) and (4.38). (i) may be proved by induction on $\Gamma \subset \Lambda$.

Now put

$$F(v) = u_{m+1}^{g_v}\left(\{k_{i,v}\}_{i=1}^{m+1}\right)(s, s') \tag{4.43}$$

As mentioned in Section 4.1, this makes sense as a function on $C(\Omega'_\Lambda)$ and is differentiable in v. Introduce the notation

$$\partial_\Gamma = \prod_{a \in \Gamma} \frac{\partial}{\partial v_a}, \qquad \Gamma \subset \Lambda \tag{4.44}$$

The order of integration or differentiation will be immaterial in what follows. We claim that

$$\left[u_{m+1}^{g}\left(\{k_i\}_1^{m+1}\right)\right]^\Lambda (s, s') = \int_{[0,1]^{|\Lambda|}} \partial_\Lambda F(v)\, dv \tag{4.45}$$

(Since the right-hand side involves derivatives of Ursell functions [cf. (4.10)], it is a linear combination of Ursell functions u_{n+m}, $n = 1, 2, \ldots, |\Lambda| + 1$. We have thus reduced the problem of estimating the left-hand side to that of the estimates of the previous section.) To see why

(4.45) is true, note that from Section 2.1, (4.42) and (4.43),

LHS (4.45)

$$= \sum_{\Gamma\subset\Lambda} (-1)^{|\Gamma|} u^{g_\Gamma}_{m+1}\big(\{k_{i,\Gamma}\}_1^{m+1}\big)$$

$$= \sum_{\Gamma\subset\Lambda} (-1)^{|\Gamma|} F(v_a = 0 \text{ if } a\in\Gamma,\ 1 \text{ otherwise})$$

$$= \sum_{\Gamma\subset\Lambda-b} (-1)^{|\Gamma|}\{F(v_a = 0 \text{ if } a\in\Gamma,\ v_a = 1 \text{ if } a\in(\Lambda-b)-\Gamma;\ v_b=1)$$

$$- F(v_a = 0 \text{ if } a\in\Gamma,\ v_a = 1 \text{ if } a\in(\Lambda-b)-\Gamma;\ v_b=0)\}$$

for some $b\in\Lambda$. But by the fundamental theorem of the calculus, this equals

$$\sum_{\Gamma\subset\Lambda-b} (-1)^{|\Gamma|}\int_0^1 dv_b \frac{\partial}{\partial v_b} F(v_a = 0 \text{ if } a\in\Gamma,\ v_a = 1 \text{ if } a\in(\Lambda-b)-\Gamma;$$

$$v_b \text{ variable}). \tag{4.46}$$

Proceeding by attrition, we arrive at (4.45).

Let us now evaluate the first few derivatives of $F(v)$. If $a\in\Lambda$ we have from (4.10) that

$$\partial_a F(v) = u_{m+2}\big(\partial_a g, \{k_i\}_1^{m+1}\big) + \sum_{j=1}^{m+1} u_{m+1}\big(\partial_a k_j, \{k_i\}_{i\neq j}\big) \tag{4.47}$$

where we are omitting mention of some indices. If $b\in\Lambda$, $b\neq a$, we have

$$\partial_{a,b}F(v) = u_{m+3}\big(\partial_a g, \partial_b t, \{k_i\}\big) + u_{m+2}\big(\partial_{a,b} g, \{k_i\}\big)$$

$$+ \sum_{j=1}^{m+1} u_{m+2}\big(\partial_a g, \partial b k_j, \{k_i\}_{i\neq j}\big)$$

$$+ \sum_j u_{m+2}\big(\partial_b g, \partial_a k_j, \{k_i\}_{i\neq j}\big)$$

$$+ \sum_j u_{m+1}\big(\partial_{a,b} k_j, \{k_i\}_{i\neq j_1}\big)$$

$$+ \sum_{j_1}\sum_{j_2\neq j_1} u_{m+1}\big(\partial_a k_{j_1}, \partial_b k_{j_2}, \{k_i\}_{i\neq j_1,\, i\neq j_2}\big) \tag{4.48}$$

In order to write a formula for $\partial_\Lambda F(v)$ for general Λ, one evidently needs some more notation. This will be discussed in Section 4.10 of this paper. In this section we will concentrate on the existence and continuity of the first two derivatives of the Gibbs state and only indicate a means of proving the full force of Proposition 4.3 and Theorem 4.4. For details see Ref. 23, Section 4.10. With this goal then, it is clear that we will need at least the following estimates of boundedness:

$$\begin{array}{lcc} & \text{First derivative} & \text{Second derivative} \\ \partial_{\psi}\zeta_j^{\phi} = u_2: & C^1 \to C^0 & C^2 \to C^1 \\ \partial_{\psi_1,\psi_2}\zeta_j^{\phi} = u_3: & — & C^2 \to C^0 \end{array} \tag{4.49}$$

For the continuity of $\phi \to \zeta_j^{\phi} f$, which in our method of proof entails the taking of one derivative, we will need also an estimate of $|u_2|_\infty$, $|u_2|_1$, and $|u_2|_2$. As we shall see, these estimates will also yield the continuity of α_1 and α_2 on the appropriate spaces. Thus, what we are looking for are estimates of

$$|u_2|_\infty,\ |u_3|_\infty,\ |u_2|_1,\ \text{and}\ |u_2|_2 \tag{4.50}$$

It is clear from (4.47) and (4.48) that estimates of $|u_2|_\infty$, $|u_3|_\infty$, and $|u_4|_\infty$ will suffice. In general, for the nth derivative, we will need estimates for $|u_l|_\infty$, where $l = 2, \ldots, n+2$.

4.7. Lemma

Ursell functions below are as in (4.7) (g and j fixed). Let f, k, k_1, k_2, and k_3 be in $C(\Omega)$ and let i, j, and l be distinct points of L. Then we have ($C^0 = C(\Omega)$):

(a) C^0 *estimates*:

$$|u_2(f,k)|_\infty \leqslant D_j(k)\min\left[|f|_\infty, D_j(f)\right] \tag{4.51}$$

$$|u_3(f,k_1,k_2)|_\infty \leqslant D_j(f)D_j(k_1)D_j(k_2) \tag{4.52}$$

$$|u_4(f,k_1,k_2,k_3)|_\infty \leqslant 4D_j(f)D_j(k_1)D_j(k_2)D_j(k_3) \tag{4.53}$$

(b) C^1 *estimates*:

$$D_l(u_2(f,k)) \leqslant D_j(f)D_{j,l}(k) + D_j(f)D_j(k)D_{j,l}(g) + D_{j,l}(f)D_j(k) \tag{4.54}$$

$$\begin{aligned} D_l(u_3(f,k_1,k_2)) \leqslant D_j(f)\big[&D_j(k_1)D_{j,l}(k_2) + D_{j,l}(k_1)D_j(k_2) \\ &+ 4D_j(k_1)D_j(k_2)D_{j,l}(g)\big] + D_{j,l}(f)D_j(k_1)D_j(k_2). \end{aligned} \tag{4.55}$$

(c) C^2 *estimate*:

$$D_{i,l}(u_2(f,k)) \leqslant D_j(f)\big[D_{j,i,l}(k) + (D_{j,l}(k)D_{j,i}(g) + D_{j,i}(k)D_{j,l}(g)) + D_j(k)D_{j,i,l}(g)\big]$$
$$+\text{same expression with } k \text{ and } f \text{ interchanged}$$
$$+4D_j(f)D_j(k)D_{j,i}(g)D_{j,l}(g) \tag{4.56}$$

Moreover, suppose that $g = k_\phi$, $k = k_\psi$, $k_i = k_{\psi_i}$ as in (4.1). Put $H^i = H_{\psi_i}$ as in (4.13). Then we have

(a′) C^0 *estimates*:

$$|u_2(f,k)|_\infty \leqslant \min\{D_j(f)\|\psi\|_1, D_j(f)|H_\psi|_1, |f|_\infty|H_\psi|_1\} \tag{4.57}$$

$$|u_3(f,k_1,k_2)|_\infty \leqslant D_j(f)|H^1|_1|H^2|_1 \tag{4.58}$$

$$|u_4(f,k_1,k_2,k_3)|_\infty \leqslant 4D_j(f)|H^1|_1|H^2|_1|H^3|_1 \tag{4.59}$$

(b′) C^1 *estimates*:

$$|u_2(f,k)|_1 \leqslant D_j(f)(|H_\psi|_2 + |H_\psi|_1|H_\phi|_2) + \sum_{\substack{l\\ l\neq j}} D_{j,l}(f)\cdot|H_\psi|_1 \tag{4.60}$$

$$|u_3(f,k_1,j_2)|_1 \leqslant D_j(f)\big(|H^1|_1|H^2|_2 + |H^1|_2|H^2|_1 + 4|H^1|_1|H^2|_1|H_\phi|_2\big)$$
$$+\sum_{\substack{l\\ l\neq j}} D_{j,l}(f)\cdot|H^1|_1|H^2|_1 \tag{4.61}$$

(c′) C^2 *estimate*:

$$|u_2(f,k)|_2 \leqslant D_j(f)\big(|H_\psi|_3 + |H_\psi|_2|H_\phi|_2 + 2|H_\psi|_1|H_\phi|_2^2 + |H_\psi|_1|H_\phi|_3\big)$$
$$+|H_\psi|_1\left[\sum_{\substack{i\\ i\neq j}} D_{j,i}(f)\cdot|H_\phi|_2 + \sum_{\substack{\{i,l\}\in V_2\\ j\notin\{i,l\}}} D_{j,i,l}(f)\right]$$
$$+|H_\psi|_2\sum_{\substack{i\\ i\neq j}} D_{j,i}(f) \tag{4.62}$$

This concludes the statement of the lemma. [In deriving the C^1 and C^2 estimates, we have only used (4.51) in the form $|u_2(f,k)|_\infty \leqslant D_j(k)D_j(f)$.]

Note that each of the estimates (4.57)–(4.62) is in fact summable over $j \in L$, given that f, ϕ, ψ etc. are in appropriate spaces.

Proof. (a) The C^0 estimates follow directly from (4.32) and (4.34) of Section 4.5, since $t_2 = t_3 = 1$ and $t_4 = 4$.

(b) For $l \neq j$, let $s, t \in \Omega_{\Lambda - j}$, with $s = t$ off l and consider the specialization of (4.45), namely,

$$[u_2(h,k)]^l = \int_0^1 dv \frac{d}{dv} F(v), \qquad v \in R^1 \tag{4.63}$$

where $F(v) = u_2^{g_v}(f_v, k_v)$ as in (4.43) and

$$g_v = vg(x_\vee s) + (1-v) g(x_\vee t) = g_l(x) + v(g(x) - g_l(x)) \tag{4.64}$$

and where $g_l(x) = g(x_\vee t)$; $g(x) = g(x_\vee s)$ as in (4.36). Similarly, $k_v = k_l + v(k - k_l)$, etc.

Thus, by (4.10),

$$\begin{aligned} \frac{dF}{dv}(v) &= u_2\left(f_v, \frac{dk_v}{dv}\right) + u_3\left(f_v, k_v, \frac{dg_v}{dv}\right) + u_2\left(\frac{df_v}{dv}, k_v\right) \\ &= u_2(f_v, k - k_l) + u_3(f_v, k_v, g - g_l) + u_2(f - f_l, k_v) \end{aligned} \tag{4.65}$$

But the Ursell function is multilinear and $0 \leqslant v \leqslant 1$, so that

$$\begin{aligned} \left|\frac{dF}{dv}\right| \leqslant & \left[v|u_2(f, k - k_l)| + (1-v)|u_2(f_l, k - k_l)|\right] \\ & + \left[v|u_2(f - f_l, k)| + (1-v)|u_2(f - f_l, k_l)|\right] \\ & + \left[v^2|u_3(f, k, g - g_l)| + v(1-v)|u_3(f, k_l, g - g_l)|\right] \\ & + \left[(1-v)v|u_3(f_l, k, g - g_l)| + (1-v)^2|u_3(f_l, k_l, g - g_l)|\right] \end{aligned} \tag{4.66}$$

Now consider, for example,

$$\begin{aligned} u_2(f_l, k - k_l) &\equiv u_2^{g_v(s,t)}(f(x_\vee t), k(x_\vee s) - k(x_\vee t)) \\ &= u_2^{g_v(s,t)}(f(x_\vee t), k^l(x_\vee s, x_\vee t)) \end{aligned} \tag{4.67}$$

where we are using the generalized sense of Ursell functions. Since $\sup_x((k^l)^j) \leqslant D_{j,l}(k)$ and $\sup_{x,y}|f(x_\vee t) - f(y_\vee t)| \leqslant D_j(f)$, we have from (4.35) that

$$|u_2(f_l, k - k_l)| \leqslant D_j(f) D_{j,l}(k) \tag{4.68}$$

as in Section 4.5. Thus we see that each bracketed term in (4.66) has the *same* estimates for the Ursell functions in it. Indeed, from part (a) of this lemma we have

$$\left|\frac{dF}{dv}\right| \leqslant D_j(f)D_{j,l}(k) + D_{j,l}(f)D_j(k) + D_j(f)D_j(k)D_{j,l}(g) \tag{4.69}$$

Note that the coefficients in v sum to unity in each case. This situation obtains for general Λ and m, and is a consequence of multilinearity and (4.39).

Integrating (4.69) over v we arrive at (4.54).

Next we prove (4.55). The above argument is modified by putting

$$F(v) = u_3^{g_v}(f_v, k_{1,v}, k_{2,v}) \tag{4.70}$$

so that

$$\left[u_3(f, k_1, k_2)\right]^l = \int_0^1 dv \, \frac{dF}{dv} \tag{4.71}$$

and now

$$\begin{aligned} \frac{dF}{dv} &= u_3(f_v, k_1 - k_{1,l}, k_{2,v}) + u_3(f_v, k_{1,v}, k_2 - k_{2,l}) \\ &\quad + u_4(f_v, k_{1,v}, k_{2,v}, g - g_l) + u_3(f - f_l, k_{1,v}, k_{2,v}) \end{aligned} \tag{4.72}$$

By a straightforward computation, we may simply drop the v's and use part (a) to arrive at (4.55).

(c) *C^2 estimate*:

We now have the variables $v = (v_i, v_l)$ so that $F(v) = u_2^{g_v}(f_v, k_v)$, where, suppressing the unchanged configurations in $\Omega_{\{i,l\}}c$,

$$\begin{aligned} k &= k(x_i, x_l), \quad k_i = k(x_i', x_l) \\ k_l &= k(x_i, x_l'), \quad \text{and} \quad k_{il} = k(x_i', x_l') \end{aligned} \tag{4.73}$$

$$k_v = v_i v_l k + (1 - v_i) v_l k_i + v_i (1 - v_l) k_l + (1 - v_i)(1 - v_l) k_{il} \tag{4.74}$$

Now if we write

$$\begin{aligned} k_l^i &= k_l - k_{il} \\ k_i^l &= k_i - k_{il} \end{aligned} \tag{4.75}$$

and

$$k^{il} = k^{\{i,l\}} = k - k_i - k_l + k_{il}$$

then we have

$$\begin{aligned} \partial_i k_v &= v_l (k - k_i) + (1 - v_l)(k_l - k_{i,l}) = v_l k^i + (1 - v_l) k_l^i \\ &= \sum_{\Gamma \subset \{l\}} K(\{l\}, \Gamma) k_\Gamma^i \qquad \left[\text{recall (4.37)}\right] \\ \partial_l k_v &= v_i k^l + (1 - v_i) k_i^l \end{aligned} \tag{4.76}$$

and

$$\partial_{il} k_v = k^{il}$$

Thus, using (4.48) we have

$$\begin{aligned}\partial_{il}F(v) = &\left[u_3(\partial_i f_v, k_v, \partial_l g_v) + u_3(\partial_l f_v, k_v, \partial_i g_v) \right] \\ &+ \left[u_3(f_v, \partial_i k_v, \partial_l g_v) + u_3(f_v, \partial_l k_v, \partial_i g_v) \right] \\ &+ \left[u_3(f_v, k_v, \partial_{il} g_v) \right] \\ &+ \left[u_4(f_v, k_v, \partial_i g_v, \partial_l g_v) \right] \\ &+ \left[u_2(f_v, \partial_{il} k_v) \right] + \left[u_2(\partial_{il} f_v, k_v) \right] \\ &+ \left[u_2(\partial_i f_v, \partial_l k_v) + u_2(\partial_l f_v, \partial_i k_v) \right] \end{aligned} \tag{4.77}$$

Now using (4.73) and (4.75) and using the ideas expressed in the proof of the first C^1 estimate above, we may see that, for example,

$$\begin{aligned} &|u_3(f_v, \partial_l k_v, \partial_i g_v) \\ &= \left| u_3\left(\sum_{\Gamma_1 \subset \{i,l\}} K(\{i,l\}, \Gamma_1) f_{\Gamma_1}, \sum_{\Gamma_2 \subset \{i\}} K(\{i\}, \Gamma_2) k^l_{\Gamma_2}, \right.\right. \\ &\qquad \left.\left. \sum_{\Gamma_3 \subset \{l\}} k(\{l\}, \Gamma_3) g^i_{\Gamma_3} \right) \right| \\ &\leqslant \sum_{\Gamma_1 \subset \{i,l\}} K(\{i,l\}, \Gamma_1) \sum_{\Gamma_2 \subset \{i\}} K(\{i\}, \Gamma_2) \sum_{\Gamma_3 \subset \{l\}} K(\{l\}, \Gamma_3) \\ &\quad \times |u_3(f_{\Gamma_1}, k^l_{\Gamma_2}, g^i_{\Gamma_3})| \end{aligned} \tag{4.78}$$

since $u_3(\cdot)$ is multilinear and each $K(\cdot, \cdot)$ is nonnegative. (We have reverted to Section 4.6-notation to give a sneak preview of why we can, in general, drop the v's.) As we have seen,

$$|u_3(f_{\Gamma_1}, k^l_{\Gamma_2}, g^i_{\Gamma_3})| \leqslant D_j(f) D_j(k^l) D_j(g^i) = D_j(f) D_{j,l}(k) D_{j,k}(g) \tag{4.79}$$

By (4.39) applied to (4.78), then,

$$|u_3(f_v, \partial_l k_v, \partial_i g_v)| \leqslant D_j(f) D_{j,l}(k) D_{j,i}(g) \tag{4.80}$$

and proceeding in this way on (4.77), we have (4.56).

We now turn to the estimates (a′), (b′), and (c′). Note firstly that $D_\Lambda(u_m) \equiv 0$ whenever $j \in \Lambda$. By (4.20), if $j \notin \Lambda$, $D_{j\cup\Lambda}(k_\psi) = \lambda^\psi(j, \Lambda)$ and thus, by (4.19),

$$\sum_{\substack{\Lambda \in V_n \\ j \notin \Lambda}} D_{j\cup\Lambda}(k_\psi) \leqslant |H_\psi|_{n+1} \tag{4.81}$$

Apply this to part (a) with $n = 0$ to get (a′), and to part (b) with $n = 0, 1$ to get (b′). In part (c) a little care with counting is well rewarded (by a factor of 2): for example,

$$\sum_{\substack{\{i,l\}\in V_2\\ j\notin\{i,l\}}} (D_{j,i}(k)D_{j,l}(g) + D_{j,l}(k)D_{j,i}(g)) \leqslant \Big(\sum_i D_{j,i}(k)\Big)\Big(\sum_l D_{j,l}(g)\Big) \tag{4.82}$$

since the sum in the left-hand side is over *unordered* pairs of distinct elements.

4.8. Partial Proof of Proposition 4.3

The left-hand sides of (4.24a) and (4.24b) are undefined unless at least one of $\alpha_n(\phi), \alpha_n(\phi')$ is finite ($n = 1, 2$ respectively). We first demonstrate (4.24a) under this assumption, for otherwise arbitrary ϕ, ϕ' in $\mathscr{P}_1$; setting $\phi' = 0$ shows then that, in fact, $\alpha_n(\phi)$ is finite whenever $\phi \in \mathscr{P}_{1,2}$. The continuity is then obvious, similarly for (4.24b).

So suppose that one of $\alpha_n(\phi), \alpha_n(\phi')$ is finite (and $\phi, \phi' \in \mathscr{P}_1$). From Section 2.6 we have

$$\begin{aligned}
|\alpha_n(\phi') - \alpha_n(\phi)| &= \tfrac{1}{2}\left|\sup_j \sum_{\Lambda\in V_n} R_{\Lambda,j}(\phi') - \sup_j \sum_{\Lambda\in V_n} R_{\Lambda,j}(\phi)\right| \\
&\leqslant \tfrac{1}{2}\sup_j \sum_{\substack{\Lambda\\ j\notin\Lambda}} \sup_{(s,t)\in\Omega'_\Lambda} \left\|\left[\mu_j^{\phi'}\right]^{\Lambda}(s,t) - \left[\mu_j^{\phi}\right]^{\Lambda}(s,t)\right\| \\
&= \tfrac{1}{2}\sup_j \sum_{\substack{\Lambda\\ j\notin\Lambda}} \sup_{(s,t)} \sup_{\substack{h\in C(X_j)\\ |h|_\infty = 1}} \left|\left[\left(\mu_j^{\phi'} - \mu_j^{\phi}\right)(h)\right]^{\Lambda}(s,t)\right|
\end{aligned} \tag{4.83}$$

(since h does not depend on $\Omega_{\{j\}^c}$, we can bring it inside the square brackets). Now

$$\left(\mu_j^{\phi'} - \mu_j^{\phi}\right)(h) = \int_0^1 du\, \frac{d}{du}\, \mu_j^{\phi+u\psi}(h) = \int_0^1 du\, u_2^f(h,k) \tag{4.84}$$

where

$$\psi = \phi' - \phi, \qquad g = k_\phi, \qquad k = k_{\phi'-\phi} \tag{4.85}$$

and

$$f = g + uk = k_{u\phi' + (1-u)\phi}$$

Hence

$$|\alpha_n(\phi') - \alpha_n(\phi)| \leqslant \tfrac{1}{2}\sup_j \sum_{\substack{\Lambda\in V_n\\ j\notin\Lambda}} \sup_h \int_0^1 du\, D_\Lambda\big(u_2^f(h,k)\big) \tag{4.86}$$

(where it is understood that u_2 is with respect to $\zeta_j^{u\phi'+(1-u)\phi}$).

We now specialize to the instances $n = 1,2$. For $n = 1$ we have

$$\sum_{\substack{l\\ l\neq j}} D_l\big(u_2^f(h,k)\big) = |u_2^f(h,k)|_1$$
$$\leqslant (\min|h|_\infty, D_j(h))|H_{\phi'-\phi}|_2 + D_j(h)|H_{\phi'-\phi}|_1|H_{u\phi'+(1-u)\phi}|_2)$$
$$\leqslant |H' - H|_2 + 2|H' - H|_1(u|H'|_2 + (1-u)|H|_2) \tag{4.87}$$

where we have used Lemma 4.7(b') with $D_j(h) \leqslant 2|h|_\infty$, $|h|_\infty = 1$, $D_{j,l}(h) = 0$ (since $h \in C(X_j)$), and with $H_\phi = H$, $H_{\phi'} = H'$. By the monotone convergence theorem applied to (4.86), (4.87) yields (4.24a).

For $n = 2$, the relevant sum in (4.86) is obtained from Lemma 4.7(c′) and (4.85), to yield (4.24b).

4.9. Partial Proof of Theorem 4.4

(a) *Continuity and Boundedness of $\zeta_j^\phi f$.* We will demonstrate that $|\zeta_j^\phi f|_N < \infty$ for $f \in C^N$, then prove continuity of $\phi \to \zeta_j^\phi f : \mathscr{P}_{N+1} \to C^N$.

For $N = 0$, we know that ζ_j^φ is a contraction on C^0 $(= C(\Omega))$ if $\varphi \in \mathscr{P}_1$. Also, $\varphi \to \zeta_j^\varphi f : \mathscr{P}_1 \to C^0$ is continuous by usual dominated convergence arguments applied to

$$\zeta_j^\varphi f(s) = Z_\varphi(s)^{-1}\int_{X_j} \nu_j(dx)\, f(x_\vee s)e^{k_\varphi(x_\vee s)} \tag{4.88}$$

using the facts that $\varphi \to k_\varphi : \mathscr{P}_1 \to C(\Omega)$ is continuous, and

$$|e^{k_\varphi} - e^{k_{\varphi'}}| \leqslant e^{|k_\varphi|_\infty}|k_\varphi - k_{\varphi'}|_\infty e^{|k_\varphi - k_{\varphi'}|_\infty} \tag{4.89}$$

Thus part (a) of the theorem is proved for $N = 0$.

Suppose $N = 1$. We have from (2.26) that

$$|\zeta_j f|_1 \leqslant \sum_{\substack{i\\ i\neq j}} D_i(f) + \sum_{\substack{i\\ i\neq j}} R_{i,j}D_j(f) \leqslant |f|_1 + \alpha_1 D_j(f) \tag{4.90}$$

proving (4.25) for $N = 1$. So suppose $N \geqslant 2$. Now we have

$$|\zeta_j f|_N \leqslant |f|_N + \sum_{m=1}^{N} \sum_{\substack{\Lambda\in V_N\\ j\notin\Lambda}} \sum_{\substack{\Gamma\in V_m\\ \Gamma\subset\Lambda}} R_{\Gamma,j}D_{(\Lambda-\Gamma)\cup j}(f) \tag{4.91}$$

which may be rewritten as

$$|\zeta_j f|_N \leqslant |f|_N + \sum_{m=1}^{N} \sum_{\substack{\Gamma\in V_m\\ j\notin\Gamma}} R_{\Gamma,j} \sum_{\substack{\Gamma'\in V_{N-m}\\ \Gamma'\cap\Gamma=\varnothing\\ j\notin\Gamma'}} D_{\Gamma'\cup j}(f) \tag{4.92}$$

If we drop the requirement $\Gamma \cap \Gamma' = \emptyset$ and use the definition of α_N we get (4.25).

Thus we have $\zeta_j^\phi f \in C^N$ and proceed to the ϕ continuity. Recalling (2.27),

$$2^{|\Lambda|}\left[\zeta_j^\phi f\right]^\Lambda(s,t)$$
$$= \sum_{\Gamma,\Gamma'\subset\Lambda}(-1)^{|\Gamma'|}\int_{X_j}\left[\mu_j^\phi\right]^\Gamma(dx\,|\,s_{\Gamma'},(s_{\Gamma'})_\Gamma)\cdot f^{\Lambda-\Gamma}(x_v\hat{s}_{\Gamma'},x_v(\hat{s}_{\Gamma'})_{\Lambda-\Gamma}) \tag{4.93}$$

with $(s,t)\in\Omega'_\Lambda$ and $j\notin\Lambda$. Using the definition (2.4) of the Nth-order difference, we write this as

$$2^{|\Lambda|}\left[\zeta_j^\phi f\right]^\Lambda(s,t) = \sum_{\Gamma,\Gamma'\subset\Lambda}(-1)^{|\Gamma'|}\sum_{\Upsilon\subset\Gamma}(-1)^{|\Upsilon|}\langle f_{\Gamma'}^{\Lambda-\Gamma}\rangle_{\phi,\Upsilon,\Gamma,\Gamma'} \tag{4.94}$$

where

$$f_{\Gamma'}^{\Lambda-\Gamma}(x) = f^{\Lambda-\Gamma}(x_v\hat{s}_{\Gamma'},x_v(\hat{s}_{\Gamma'})_{\Lambda-\Gamma}) \tag{4.95}$$

$$\langle\cdot\rangle_{\phi,\Upsilon,\Gamma,\Gamma'} = \int_{X_j}\mu_j^\phi(dx\,|\,(s_{\Gamma'})_\Upsilon) \tag{4.96}$$

In (4.96) it is to be understood that the symbol $(s_{\Gamma'})_\Upsilon$ refers to the $|\Gamma|$th-order difference symbol of $s_{\Gamma'}$ on the pair $(s_{\Gamma'},(s_{\Gamma'})_\Gamma)$, which in turn refers to the pair (s,t). This is why we retain the subscript Γ in the left-hand side of (4.96).

Now select any one triple Γ,Γ',Υ. We will, whenever convenient, suppress some indices. Consider $u\to\langle f_{\Gamma'}^{\Lambda-\Gamma}\rangle_{\phi+u\psi,\Upsilon,\Gamma,\Gamma'}$ with a view towards differentiating (4.94). To this end, put

$$k_{\Gamma'} = k_\psi(x_v s_{\Gamma'}) \tag{4.97}$$

and

$$k_{\Gamma';\Upsilon} = k_\psi(x_v(s_{\Gamma'})_\Upsilon) \qquad (k_\phi = (x_v s) = k)$$

where, again, the switching is with reference to $(s_{\Gamma'},(s_{\Gamma'})_\Gamma)$.

Then by (4.3),

$$\frac{d}{du}\langle f_{\Gamma'}^{\Lambda-\Gamma}\rangle_{\phi+u\psi,\Upsilon,\Gamma,\Gamma'} = \int_{X_j}\mu_j^{\phi+u\psi}(dx\,|\,(s_{\Gamma'})_\Upsilon)\cdot f_{\Gamma'}^{\Lambda-\Gamma}$$
$$\cdot\left(k_{\Gamma',\Upsilon} - \langle k_{\Gamma',\Upsilon}\rangle_{\phi+u\psi,\Upsilon,\Gamma,\Gamma'}\right)$$
$$= u_2^{\phi+u\psi}\left[k_{\Gamma'},f_{\Gamma'}^{\Lambda-\Gamma}(s_{\Gamma'},(s_{\Gamma'})_{\Lambda-\Gamma})\right]((s_{\Gamma'})_\Upsilon) \tag{4.98}$$

where we hope the notation is clear. It emphasizes that the argument $f_{\Gamma'}^{\Lambda-\Gamma}$ is independent of Υ, because our intention is to differentiate (4.94) and write the resulting sum over $\Upsilon \subset \Gamma$ there as a $|\Gamma|$th-order difference. From (4.94), (4.96), and (4.98) we have

$$
\begin{aligned}
&2^{|\Lambda|}\left[\frac{d}{du}\zeta_j^{\phi+u\psi}f\right]^{\Lambda}(s,t)\\
&\quad=\sum_{\Gamma,\Gamma'\subset\Lambda}(-1)^{|\Gamma'|}\sum_{\Upsilon\subset\Gamma}(-1)^{|\Upsilon|}u_{2,\Upsilon,\Gamma,\Gamma'}^{\phi+u\psi}\left(k_{\Gamma'},f_{\Gamma'}^{\Lambda-\Gamma}(s_{\Gamma'},(s_{\Gamma'})_{\Lambda-\Gamma})\cdot((s_{\Gamma'})_{\Upsilon}\right)\\
&\quad=\sum_{\Gamma,\Gamma'\subset\Lambda}(-1)^{|\Gamma'|}\left[\left[u_2^{\phi+u\psi}\left(k_{\Gamma'},f_{\Gamma'}^{\Lambda-\Gamma}(s_{\Gamma'},(s_{\Gamma'})_{\Lambda-\Gamma})\right]\right]^{\Gamma}\cdot(s_{\Gamma'},(s_{\Gamma'})_{\Gamma})
\end{aligned}
\tag{4.99}
$$

Here the double brackets emphasize that the argument of $f_{\Gamma'}^{\Lambda-\Gamma}$ is unaffected in the taking of the $|\Gamma|$th-order difference. The advantage of worrying about this is that when we use Lemma 4.7 in estimating u_2, all terms like $D_{j,l}(f_{\Gamma'}^{\Lambda-\Gamma}), D_{j,l,k}(f_{\Gamma'}^{\Lambda-\Gamma})$ are automatically spiflicated.

The next step is to use

$$
\begin{aligned}
|\zeta_j^{\phi'}f-\zeta_j^{\phi}f|_N&=\sum_{\Lambda\in V_N}D_\Lambda\left(\int_0^1 du\,\frac{d}{du}\zeta_j^{\phi+u\psi}f\right)\\
&\leqslant\sum_{\Lambda\in V_N}\sup_{\Omega'_\Lambda}\int_0^1 du\left|\left[\frac{d}{du}\zeta_j^{\phi+u\psi}f\right]^{\Lambda}\right|(\cdot)
\end{aligned}
\tag{4.100}
$$

That is, from (4.99),

$$
|\zeta_j^{\phi}f-\zeta_j^{\phi'}f|_N\leqslant 2^{-N}\int_0^1 du\sum_{\Lambda\in V_N}\sup_{\Omega'_\Lambda}\sum_{\Gamma\subset\Lambda}\sum_{\Gamma'\subset\Lambda}\left|\left[\left[u_2^{\phi+u\psi}(k_{\Gamma'},f_{\Gamma'}^{\Lambda-\Gamma})\right]\right]^{\Gamma}\right|
\tag{4.101}
$$

where $\psi=\phi'-\phi$.

We specialize now to the instances $N=1,2$.

(i) $|\Lambda|=1$. For $\Lambda=\{l\}$ $(l\neq j)$, (4.99) gives

$$
\begin{aligned}
2\left|\left[\frac{d}{du}\zeta_j^{\phi+u\psi}f\right]^{l}\right|&\leqslant\sum_{\Gamma'\subset\{l\}}\left\{|u_2^{\phi+u\psi}(f_{\Gamma'}^{l},k_{\Gamma'})|+\left|\left[\left[u_2^{\phi+u\psi}(f_{\Gamma'}^{\varnothing},k_{\Gamma'})\right]\right]^{l}\right|\right\}\\
&\leqslant\sum_{\Gamma'\subset\{l\}}\left[D_l(f)D_j(k)+D_j(f)(D_{j,l}(k)+D_j(k)D_{j,l}(g))\right]
\end{aligned}
\tag{4.102}
$$

where we have used (4.51) and (4.54) and the arguments leading to them (with $D_{j,l}(f)$ being set equal to zero.) Here $g=k_{\phi+u\psi}$. Now put $\psi=\phi'-\phi$, $H=H_\phi$, $H'=H_{\phi'}$. Then by (4.20)

$$
D_j(k)=|H'-H|_1,\qquad \sum_l D_{j,l}(k)=|H-H'|_2
\tag{4.103}
$$

and

$$\sum_l D_{j,l}(g) \leqslant u|H'|_2 + (1-u)|H|_2$$

since $0 \leqslant u \leqslant 1$.

Now from (4.101), (4.102), and (4.103) we get

$$|\zeta_j^{\phi'}f - \zeta_j^{\phi}f|_1 \leqslant |H' - H|_1|f|_1 + D_j(f)\big[|H - H'|_2 + \tfrac{1}{2}|H - H'|_1(|H'|_2 + |H|_2)\big] \quad (4.104)$$

so that $\phi \to \zeta_j^{\phi}f : \mathscr{P}_2 \to C^1$ is continuous for each $f \in C^1$.

(ii) $|\Lambda| = 2$. For $\Lambda = \{i, l\}$, $j \notin \Lambda$, (4.99) gives

$$4\left|\left[\frac{d}{du}\zeta_j^{\phi+u\psi}f\right]^{i,l}\right| \leqslant \sum_{\Gamma' \subset \Lambda} |u_2^{\phi+u\psi}(f_{\Gamma'}^{i,l}, k_{\Gamma'})| + \left|\big[\big[u_2(f_{\Gamma'}^{l}, k_{\Gamma'})\big]\big]^{i}\right| + \left|\big[\big[u_2(f_{\Gamma'}^{i}, k_{\Gamma'})\big]\big]^{l}\right| + \left|\big[\big[u_2(f_{\Gamma'}, k_{\Gamma'})\big]\big]^{i,l}\right| \quad (4.105)$$

Again, the sum over Γ' cancels the coefficient 4 on the left-hand side and summing over $\{i, l\}$ using Lemma 4.7 (and setting $D_{j,i}(f^l) = 0$ etc. there) we have from (4.103) that

$$\left|\frac{d}{du}\tau_j^{\varphi+u\psi}f\right|_2 \leqslant |f|_2|H - H'|_1 + \left(\sum_{\substack{l \\ l \neq j}} D_{j,l}(f)\right) \times \big[|H - H'|_2 + |H - H'|_1(u|H'|_2 + (1-u)|H|_2)\big]$$
$$+ D_j(f)\Big[|H - H'|_3 + |H - H'|_2(u|H'|_2 + (1-u)|H|_2) + |H - H'|_1\big(4(u|H'|_2 + (1-u)|H|_2)^2 + u|H'|_3 + (1-u)|H|_3\big)\Big] \quad (4.106)$$

Again, this integrates in (4.101) to show that $\phi \to \zeta_j^{\phi}f : \mathscr{P}_3 \to C^2$ is continuous for each $f \in C^2$.

(b) (i) $C(\Omega)$ ***Existence and Continuity of Derivatives.*** Existence is the content of Section 4.1. Their continuity as functions from $\mathscr{P}_1$ into $C(\Omega)$ for $f \in C(\Omega)$) follows directly from their representation as Ursell functions in (4.8) and (4.7) and part (a)(iii) of this theorem.

(b) (ii) C^N ***Boundedness and Continuity of the Derivatives.*** In this partial proof we restrict ourselves to the following cases:

(i) $\partial_{\psi}\zeta_j^{\phi}f : \phi, \psi \in \mathscr{P}_1, \quad f \in C(\Omega)$

(ii) $\partial_{\psi_1,\psi_2}\zeta_j^{\phi}f : \phi, \psi \in \mathscr{P}_2, \quad f \in C^1$

(iii) $\partial_{\psi_1,\psi_2}\zeta_j^{\phi}f : \phi, \psi \in \mathscr{P}_1, \quad f \in C(\Omega)$

Now

$$\partial_\psi \zeta_j^\phi f = u_2^\phi(f,k) \quad \text{and} \quad \partial_{\psi_1\psi_2}\zeta_j^\phi f = u_3\phi(f,k_1,k_2) \tag{4.107}$$

Hence (4.28), (4.29), and (4.30) follow directly from Lemma 4.7: (4.57), (4.58), and (4.60), respectively, and the relative Hamiltonian formulas (4.20), (4.21). Boundedness, given the pertinent locations of ϕ, f, ψ, etc., is assured ($\sum_{l\neq j} D_{j,l}(f) \leqslant 2|f|_1$, etc.).

[Actually, given the remaining estimates in Lemma 4.7, we have more:

$$\begin{aligned}
\partial_{\psi_1\psi_2\psi_3}\zeta_j^\phi f \in C(\Omega) &\quad \text{for} \quad \phi,\psi_1,\psi_2,\psi_3 \in \mathscr{P}_1 \quad \text{and} \quad f \in C(\Omega) \\
\partial_{\psi_1\psi_2}\zeta_j^\phi f \in C^1 &\quad \text{for} \quad \phi,\psi_1,\psi_2 \in \mathscr{P}_2 \quad \text{and} \quad f \in C^1 \\
\partial_{\psi}\zeta_j^\phi f \in C^2 &\quad \text{for} \quad \phi,\psi \in \mathscr{P}_3 \quad \text{and} \quad f \in C^2
\end{aligned} \tag{4.108}$$

Indeed the *number* of derivatives does not determine the requirements on f and the interactions. This is borne out in the general case by b(ii) of Theorem 4.4. By contrast, the number of derivatives of T_ϕ *will* determine these requirements.]

We turn now to continuity in ϕ of the derivatives. Continuity from $\mathscr{P}_1$ to $C(\Omega)$ needs no further proof, since Ursell functions are automatically $C(\Omega)$ continuous.

Now, we saw in Section 2.5 that for $\psi \in \mathscr{P}_{N+1}$, $k_\psi \in C^N$. Thus by the representation of derivatives of $\zeta_j^\phi f$ as Ursell functions as in (4.7), continuity from P_{N+1} to C^N follows from part (a)(iii) of this theorem.

This concludes the partial proof.

4.10. On the General Estimates and Completion of the Proofs of Proposition 4.3 and Theorem 4.4

Let us first review the extent to which we have established Proposition 4.3 and then state what we need in order to complete its proof.

(i) *Proof of Proposition 4.3.* The basic inequality required is (4.86). Suppose we can find some estimate

$$|u_2^j(h,k)|_n \leqslant \eta(\phi,\phi',u) \tag{4.109}$$

independent of $j \in L$, $h \in C(X_j)$, $|h|_\infty = 1$ and such that $\eta(\phi,\phi',u)$ is a linear combination of $|H_\phi - H_{\phi'}|_m$, $1 \leqslant m \leqslant n+1$, with coefficients possibly involving $|H_\phi|_m, |H_{\phi'}|_m$, $1 \leqslant m \leqslant n+1$ and polynomials in u. Then we will have established Proposition 4.3.

But this is quite straightforward given the formalism developed in

Section 4.6. Recalling (4.43), for $\Lambda \in V_n$ and $(s,s') \in \Omega'_\Lambda$, we have

$$F(v) = u_2^{f_v}(h, k_v), \qquad v = \{v_a\}_{a\in\Lambda} \tag{4.110}$$

where

$$f_v = \sum_{\Gamma\subset\Lambda} K(\Lambda,\Gamma) f_\Gamma \tag{4.111}$$

$$f_\Gamma(x) = f(x_v \hat{s}_\Gamma)$$

and similarly for k_v. Note that $h_v = h$, as h does not depend upon spins outside j. For $\Gamma \subset \Lambda$, let Q_Γ run over (unordered) partitions of Γ. Then we claim that

$$\partial_\Lambda F(v) = u_2^{f_v}(h, \partial_\Lambda k_v) + \sum_{\substack{\Gamma\subset\Lambda\\ \Gamma\neq\emptyset}} \sum_{Q_\Gamma} u_{2+|Q_\Gamma|}^{f_v}\left(h, \partial_{\Lambda-\Gamma} k_v, \{\partial_P f_v\}_{P\in Q_\Gamma}\right) \tag{4.112}$$

This may be proved by induction, by first replacing Λ above with $\Upsilon \subset \Lambda$ $(v = \{v_a\}_{a\in\Lambda})$ and using (4.10) [recall (4.47) and (4.48)].

Now we may generalize (4.76) above for the derivatives of $v \to k_v$ by defining, for Γ and Γ' disjoint subsets of Λ,

$$k_{\Gamma'}^{\Gamma} = k^{\Gamma}(x_v \hat{s}_{\Gamma'}, x_v \hat{s}_{\Gamma'\cup\Gamma}), \qquad \Gamma \cap \Gamma' = \emptyset \tag{4.113}$$

Then we may write any derivatives as

$$\partial_\Gamma k_v = \sum_{\Gamma'\subset\Lambda-\Gamma} K(\Lambda-\Gamma, \Gamma') g_{\Gamma'}^{\Gamma}, \qquad \Gamma \subset \Lambda \tag{4.114}$$

This may be proved (Ref. 23) by induction on Γ, using (4.40) and Proposition 2.2(c). Thus we have, returning to (4.112), that $\partial_\Lambda k_v = k^\Lambda$, whereas,

$$\partial_{\Lambda-\Gamma} k_v = \sum_{\Gamma_1\subset\Gamma} K(\Gamma, \Gamma_1) k_{\Gamma_1}^{\Lambda-\Gamma} \tag{4.115}$$

and

$$\partial_P f_v = \sum_{\Gamma_2\subset\Lambda-P} K(\Lambda-P, \Gamma_2) f_{\Gamma_2}^P \tag{4.116}$$

Applying these to (4.112) we have

$$\begin{aligned}\partial_\Lambda F(v) &= u_2^{f_v}(h, k^\Lambda)\\ &\quad + \sum_{\substack{\Gamma\subset\Lambda\\ \Gamma\neq\emptyset}} \sum_{Q_\Gamma} \left[\sum_{\Gamma_1\subset\Gamma} K(\Gamma,\Gamma_1)\right]\left[\prod_{P\in Q_\Gamma} \sum_{\Gamma_2\subset\Lambda-P} K(\Lambda-P,\Gamma_2)\right]\\ &\quad \times u_{2+|Q_\Gamma|}^{f_v}\left(h, k_{\Gamma_1}^{\Lambda-\Gamma}, \{f_{\Gamma_2}^P\}_{P\in Q_\Gamma}\right)\end{aligned} \tag{4.117}$$

We may now estimate $\partial_\Lambda F(v)$ by employing (4.32), which says here that $(D_j(h) \leqslant 2|h|_\infty \leqslant 2)$

$$\left|u^{f_\varphi}_{2+|Q_\Gamma|}\left(h, k_{\Gamma_1}^{\Lambda-\Gamma}, \left\{f_{\Gamma_2}^{P}\right\}_{P \in Q_\Gamma}\right)\right| \leqslant 2t_{2+|Q_\Gamma|} D_{j\cup(\Lambda-\Gamma)}(k) \prod_{P \in Q_\Gamma} D_{j \cup P}(f) \tag{4.118}$$

i.e., independently of Γ_1 and Γ_2. Since each of the square-bracketed terms in (4.117) involving $K(\cdot,\cdot)$ is equal to unity as in (4.39), we have that

$$\begin{aligned} D_\Lambda(u_2(h,k)) &\leqslant D_{j\cup\Lambda}(k) + 2\sum_{\substack{\Gamma\subset\Lambda\\ \Gamma\neq\varnothing}} \sum_{Q_\Gamma} t_{2+|Q_\Gamma|} D_{j\cup(\Lambda-\Gamma)}(k) \prod_{P\in Q_\Gamma} D_{j\cup P}(f) \\ &\leqslant \lambda^{\phi'-\phi}(j,\Lambda) \\ &\quad + 2\sum_{\substack{\Gamma\subset\Lambda\\ \Gamma\neq\varnothing}} \lambda^{\phi'-\phi}(j,\Lambda-\Gamma) \sum_{Q_\Gamma} t_{2+|Q_\Gamma|} \prod_{P\in Q_\Gamma} \lambda^{u\phi'+(1-u)\phi}(j,P) \end{aligned} \tag{4.119}$$

where we have used (4.45), (4.75), and (4.20). Of course, if $j \in \Lambda$ then $D_\Lambda(u_2(h,k)) = 0$. Now we wish to sum over $\Lambda \in V_n$, $j \notin \Lambda$ to get an estimate as in (4.109). This is worked out in Ref. 25, pp. 106 and 107; here we present the result. *A partition of the number N* of length p is a collection $\mathscr{T}_p^{(N)} = \{\xi_1, \ldots, \xi_p\}$ of strictly positive integers such that $N = \sum_{i=1}^{p} \xi_i$. We have then that (4.119) sums to

$$\begin{aligned} |u_2(h,k)|_n &\leqslant |H_\varphi - H_{\varphi'}|_{n+1} \\ &\quad + 2\sum_{m=1}^{n} |H_\varphi - H_{\varphi'}|_m \left[\sum_{p=1}^{n-m+1} t_{2+p} \sum_{\mathscr{T}_p^{(n-m+1)}} \right. \\ &\quad \left. \times \prod_{\xi\in\mathscr{T}_p^{(n-m+1)}} |H_{n\varphi+(1-u)\varphi'}|_{\xi+1}\right] \end{aligned} \tag{4.120}$$

Applying this to (4.86), we have proved Proposition 4.3.

(ii) *Proving Theorem 4.4.* What remains to be done in the proof of the Theorem? In part (a), we need to prove continuity of $\varphi \to \zeta_j^\varphi f : \mathscr{P}_{N+1} \to C^N$. In part (b), we have to establish (ii), which is a question of proving (4.27). For once we have that estimate, the nth derivative is in C^N for $\varphi, \psi_i \in \mathscr{P}_{N+1}$. The continuity result of part (a) then serves to complete the proof.

It is clear from Section 4.9 that what we need, in order to prove continuity and boundedness as above, is an estimate generalizing those of Lemma 4.7 and (4.120). Indeed, with the nth derivative of ζ_j replaced by the $(n+1)$th Ursell function, (4.27) is one such estimate and may be found by a simple though tedious extension of the arguments in part (i) of this subsection (see Ref. 23, pp. 107–114).

5. CONTINUOUS DIFFERENTIABILITY OF THE GIBBS STATE

In the following $\mathscr{D}_N$ is the Nth Dobrushin uniqueness region defined in Section 3.1, the spaces C^N are defined in Section 2.5 and the $\mathscr{P}_N$ in Section 1. Note the relationship (4.19) between the nth relative Hamiltonian norm and the $\mathscr{P}_n$ norm.

5.1. Theorem

Let $\phi \in \mathscr{D}_N$ and $\sigma_\phi(\cdot)$ be the unique Gibbs state corresponding to ϕ. For each $f \in C^N$, the function $\phi \to \sigma_\phi(f): \mathscr{D}_N \to R^1$ is N times continuously Gateaux differentiable in $\mathscr{P}_{N+1}$ directions. That is, for each n with $1 \leqslant n \leqslant N$, and for each collection $\{\psi_1\}_{i=1}^n$, $\psi_i \in \mathscr{P}_{N+1}$, $1 \leqslant i \leqslant n$,

$$(\partial_{\psi_1,\ldots,\psi_n^\sigma})_\phi(f) = \frac{\partial^n}{\partial u_1 \ldots \partial u_n} \sigma_{\phi+\sum_{i=1}^n u_i\psi_i}(f)|_{u_1=\cdots=u_m}=0 \qquad (5.1)$$

exists and is continuous on $\mathscr{D}_N$.

Moreover, under these conditions, the expression (5.1) is bounded in absolute value by a polynomial in the variables

$$|f|_M,\ \alpha_M(\phi),\ |H_{\psi_i}|_M \text{ and } |H_\phi|_M, \qquad 1 \leqslant M \leqslant N$$

and

$$\frac{1}{1-\alpha_1(\phi)},\ |H_{\psi_i}|_{N+1},\ \text{and } |H_\phi|_{N+1}$$

where the H and $|H|_M$ are as in (4.10) and (4.12).

The derivative is given by the convergent sum

$$(\partial_{\psi_1,\ldots,\psi_n}\sigma)_\phi(f) \equiv \partial_{\{\psi_i\}_1^n}\sigma_\phi(f)$$

$$= \sum_{Q_n}{}' \sigma_\phi\Big(\Big\{\prod_{P\in Q_n}{}' \left(\partial_P T_\phi(I-T_\phi)^{-1}\right)\Big\}[f]\Big) \qquad (5.2)$$

where: the primed sum is over ordered partitions of the set $\{1,2,\ldots,n\}$,

$$\partial_P T_\phi g = \partial_{\{\psi_i\}_{i\in P}} T_\phi g \tag{5.3}$$

(whose existence and continuity is the content of Lemma 5.4), $[f]$ is the equivalence class modulo constants of f in C^N, $(I-T_\phi)^{-1}$ is the inverse of $I-T_\phi$ acting on $\tilde{C}^N$ and is given by

$$(I-T_\phi)^{-1}[f] = \sum_{k=0}^{\infty}\left[T_\phi^k f\right] \tag{5.4}$$

and finally, where the primed product represents an ordered product: if $Q_n = (P_1,\ldots,P_m)$ is an ordered partition, then

$$\prod_{P\in Q_n}' \left(\partial_P T_\phi (I-T_\phi)^{-1}\right)$$
$$= \partial_{P_1} T_\phi (I-T_\phi)^{-1} \partial_{P_2} T_\phi (I-T_\phi)^{-1} \cdots \partial_{P_m} T_\phi (I-T_\phi)^{-1}$$

5.2. Bounds on the First and Second Derivatives

In the cases $N=1,2$, the expression (5.2) becomes

$$\partial_\psi \sigma_\phi(f) = \sigma_\phi\left(\partial_\psi T_\phi \cdot (I-T_\phi)^{-1}[f]\right) = \sum_{k=0}^{\infty} \sigma_\phi\left(\partial_\psi T_\phi \cdot T_\phi^k f\right) \tag{5.5}$$

$$\begin{aligned}\partial_{\psi_1\psi_2}\sigma_\phi(f) &= \sigma_\phi\left(\partial_{\psi_1} T_\phi \cdot (I-T_\phi)^{-1} \cdot \partial_{\psi_2} T_\phi \cdot (I-T_\phi)^{-1}[f]\right)\\ &\quad + \sigma_\phi\left(\partial_{\psi_2} T_\phi \cdot (I-T_\phi)^{-1} \cdot \partial_{\psi_1} T_\phi (I-T_\phi)^{-1}[f]\right)\\ &\quad + \sigma_\phi\left(\partial_{\psi_1\psi_2} T_\phi (I-T_\phi)^{-1}[f]\right)\\ &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\left\{\sigma_\phi\left(\partial_{\psi_1} T_\phi \cdot T_\phi^k \cdot \partial_{\psi_2} T_\phi \cdot T_\phi^l f\right)\right.\\ &\quad \left.+ \sigma_\phi\left(\partial_{\psi_2} T_\phi \cdot T_\phi^k \cdot \partial_{\psi_2} T_\phi \cdot T_\phi^l f\right)\right\} + \sum_{k=0}^{\infty} \sigma_\phi\left(\partial_{\psi_1\psi_2} T_\phi \cdot T_\phi^k f\right)\end{aligned} \tag{5.6}$$

The second equality in each case will be seen to be true in the proof of Theorem 5.1. The series in (5.5) was provided by L. Gross,[3] who also showed that

$$|\partial_\psi \sigma_\phi(f)| \leqslant \frac{\|\psi\|_1 |f|_1}{\left[1-\alpha_1(\phi)\right]^2} \tag{5.7}$$

Here we shall also explicitly demonstrate that, with $H^i = H_{\psi_i}$, $i = 1, 2$, and $\alpha_1 = \alpha_1(\phi)$, $\alpha_2 = \alpha_2(\phi)$,

$$
\begin{aligned}
&|\partial_{\psi_1\psi_2}\sigma_\phi(f)| \\
&\leqslant \Bigg(\frac{1}{2(1-\alpha_1)^2} |H^1|_1 |H^2|_1 |f|_1 \\
&\quad + \frac{\|\psi_1\|_1}{(1-\alpha_1)^2} \Bigg\{ \frac{1}{(1-\alpha_1)} (|H^2|_2 + |H^2|_1 |H_\phi|_2) |f|_1 \\
&\qquad + \frac{1}{1-\alpha_1} |H^2|_1 \Bigg[2|f|_2 + \frac{2\alpha_2}{(1-\alpha_1)^2} |f|_1 \\
&\qquad\qquad + \alpha_2\left(1 + \frac{2}{1-\alpha_1}\right)|f|_1 \Bigg] + 2\|\psi_2\|_1 |f|_1 \Bigg\} \\
&\quad + \frac{\|\psi_1\|_1}{(1-\alpha_1)^4} \Bigg\{ (|H^2|_2 + |H^2|_1 |H_\phi|_2)|f|_1 + |H^2|_1 \\
&\qquad \times \Bigg[\alpha_2\left(1 + \frac{2}{1-\alpha_1}\right)|f|_1 + 2|f|_2 + \frac{\alpha_2}{(1-\alpha_1)^2}|f|_1 \Bigg] \Bigg\} \Bigg) \\
&\quad + \{\text{same expression with } \psi_1 \leftrightarrow \psi_2 \text{ and } H^1 \leftrightarrow H^2\} \\
&\leqslant \Bigg(\frac{2}{(1-\alpha_1)^2} \|\psi_1\|_1 \|\psi_2\|_1 |f|_1 + \frac{\|\psi_1\|_1}{(1-\alpha_1)^2} \\
&\quad \times \Bigg[\frac{4}{1-\alpha_1} (\|\psi_2\|_2 + 2\|\psi_2\|_1 \|\phi\|_2)|f|_1 + \frac{2}{1-\alpha_1} \|\psi_2\|_1 (2|f|_2 \\
&\quad + \frac{2\alpha_2}{(1-\alpha_1)^2}|f|_1 + \alpha_2\left(1 + \frac{2}{1-\alpha_1}\right)|f|_1 + 2\|\psi_2\|_1 |f|_1 \Bigg] \\
&\quad + \frac{\|\psi_1\|_1}{(1-\alpha_1)^4} \Bigg\{ 4(\|\psi_2\|_2 + 2\|\psi_2\|_1 \|\phi\|_2)|f|_1 \\
&\qquad + 2\|\psi_2\|_1 \Bigg[\alpha_2\left(1 + \frac{2}{1-\alpha_1}\right)|f|_1 + 2|f|_2 \\
&\qquad\qquad + \frac{\alpha_2}{(1-\alpha_1)^2}|f|_1 \Bigg] \Bigg\} \Bigg) \\
&\quad + \{\text{same expression with } \psi_1 \leftrightarrow \psi_2\} \qquad (5.8)
\end{aligned}
$$

(where we have used the facts that $|H_\psi|_1 \leqslant 2\|\psi\|_1$ and $|H_\psi|_2 \leqslant 4\|\psi\|_2$ [see (4.19)]). Indeed, the estimates of the following lemmas along with the series expansions for the deviatives provide us with a systematic procedure for estimating the size of any derivative of the Gibbs state. However, a notation appropriate to an arbitrary number of derivatives awaits full development.

5.3. Lemma: The Boundedness and Continuity of T_ϕ

For each $k \in [1, \infty)$ and $p \in [k, \infty)$, the operator $T_{\phi,k,p}$ is a contraction on $C(\Omega)$ for $\phi \in \mathscr{P}_1$. Also, $\phi \to T_{\phi,k,p}f$ is continuous for each $f \in C(\Omega)$.

Let $N = 1, 2, \ldots$. Then for $\phi \in \mathscr{D}_N$, $T_{\phi,k,p}$ is a bounded linear operator on C^N. Moreover, for each $f \in C^N$, the function $\phi \to T_{\phi,k,p}f : \mathscr{D}_N \to C^N$ is continuous.

(See Theorem 3.2 for a bound on $|T_{\phi,k,p}f|_N$ and Remark 3.3(c) for the instances $N = 1$, 2, and 3.)

Proof. The contractivity on $C(\Omega)$ was shown in Section 1.2. For $p < \infty$, $\phi \to T_{\phi,k,p}f : \mathscr{P}_1 \to C(\Omega)$ is continuous since $T_{\phi,k,p}$ is a finite product of ϕ-continuous operators (Section 4.1). But $T_{\phi,k,p}f$ converges uniformly (in ϕ) to $T_{\phi,k,\infty}f$, in the space $C(\Omega)$. Thus continuity is assured for each $p \leqslant \infty$.

Let now $N \geqslant 1$. By Theorem 3.2, $T_{\phi,k,p}$ is a bounded linear operator on C^N. If $p < \infty$, Theorem 4.4(a) assures us that $\zeta_p^\phi f \in C^N$ for $f \in C^N$ and $\phi \in \mathscr{D}_N \subset \mathscr{P}_{N+1}$ and moreover that $\phi \to \zeta_p^\phi f : \mathscr{D}_N \to C^N$ is continuous. Thus the same holds for $\zeta_{p-1}^\phi(\zeta_p^\phi f) = T_{\phi,p-1,p}f$. In other words, $T_{\phi,k,p}f$ is continuous into C^N whenever $p < \infty$.

If g is a continuous cylinder function, there is some $q < \infty$ such that

$$T_{\phi,k,\infty}\, g = \begin{cases} T_{\phi,k,q}g & \text{if} \quad k \leqslant q \\ g & \text{if} \quad k > q \end{cases} \tag{5.9}$$

Hence in this case $T_{\phi,k,p}g$ is continuous, for all $p \leqslant \infty$, from $\mathscr{D}_N$ into C^N. Let $\{g_n\}$ be a sequence of cylinder functions tending to f in C^N, i.e., $|f - g_n|_M \to 0$ for $1 \leqslant M \leqslant N$. Put

$$B_r^N = \left\{ \phi \in \mathscr{P}_{N+1} : \alpha_1(\phi) \leqslant r \text{ and } \alpha_M(\phi) \leqslant \frac{1}{1-r}, \ 1 \leqslant M \leqslant N \right\} \tag{5.10}$$

Then $\mathscr{D}_N = \bigcup_{r<1} B_r^N$. We see from (3.2) and its adjuncts (3.3), (3.4), and (3.6) that the bound given in the right-hand side of (3.2) is uniform in ϕ for $\phi \in B_r^N$, $r < 1$. Moreover, it involves only those $|f|_M$ for which $1 \leqslant M \leqslant N$. Set $p = \infty$ so that $E^p \equiv 0$. Then it is clear that $|T_{\phi,k,\infty}f - T_{\phi,k,\infty}g_n|_M \to 0$ for each $M \leqslant N$, and uniformly in $\phi \in B_r^N$. Thus $\phi \to T_{\phi,k,\infty}f : B_r^N \to C^N$ is continuous. Finally, let $r \uparrow 1$.

5.4. Lemma: Existence, Boundedness, and Continuity of Derivatives of T_ϕ

We will sometimes suppress the symbol ϕ in the equations that follow. Also, write $T = T_{1,\infty}$, $T_k = T_{k,\infty}$.

(a) If n is a natural number, $\phi \in \mathscr{D}_n$, $\psi_i \in \mathscr{P}_{n+1}$ $(1 \leqslant i \leqslant n)$ and $f \in C^n$, the function

$$(u_1, \ldots, u_n) \to T_{\phi + \sum_i u_i \psi_i} f : R^n \to C(\Omega) \tag{5.11}$$

possesses the nth mixed derivative

$$(\partial_{\psi_1, \ldots, \psi_n} T)_\phi f = \frac{\partial^n}{\partial u_1 \ldots \partial u_n} T_{\phi + \sum_i u_i \psi_i} f|_{u_1 = \cdots = u_n = 0} \tag{5.12}$$

Moreover, this derivative is, for fixed f and $\{\psi_i\}$, continuous as a function of ϕ from $\mathscr{D}_n$ into C^n.

In fact, we may describe the derivative by a $C(\Omega)$ convergent sum as follows:

$$\partial_\psi T f = \sum_{j=1}^{\infty} T_{1,j-1} \cdot (\partial_\psi \zeta_j)(T_{j+1} f) \tag{5.13}$$

$$\begin{aligned}\partial_{\psi_1,\psi_2} T f = {} & \sum_{j=1}^{\infty} T_{1,j-1} \cdot (\partial_{\psi_1 \psi_2} \zeta_j)(T_{j+1} f) \\ & + \sum_{j=1}^{\infty} \sum_{k>j} \{ T_{1,j-1} \cdot (\partial_{\psi_1} \zeta_j)[T_{j+1,k-1}(\partial_{\psi_2} \zeta_k)(T_{k+1} f)] \\ & + T_{1,j-1}(\partial_{\psi_2} \zeta_j)[T_{j+1,k-1}(\partial_{\psi_1} \zeta_k)(T_{k+1} f)]\}\end{aligned} \tag{5.14}$$

where, as usual, $T_{l,l-1} \equiv I$. For the general case, let $Q = \{P_1, \ldots, P_s\}$ be some ordered partition of $\{1, 2, \ldots, n\}$. Define

$$\zeta_j^P f = \partial_{\{\psi_i\}_{i \in P}} \zeta_j^\phi f, \qquad P \subset \{1, \ldots, n\} \tag{5.15}$$

$$\begin{aligned}T^Q f = {} & \sum_{j_s = s}^{\infty} \sum_{j_{s-1} < j_s}^{\infty} \cdots \sum_{j_1 < j_2} T_{1,j_1 - 1} \cdot \zeta_{j_1}^{P_1} T_{j_1 + 1, j_2 - 1} \\ & \cdot \zeta_{j_2}^{P_2} \cdots T_{j_{s-1}+1, j_s - 1} \cdot \zeta_{j_s}^{P_s} T_{j_s + 1} f\end{aligned} \tag{5.16}$$

Then we have that

$$\partial_{\psi_1, \ldots, \psi_n} T_\phi f = \sum_{Q_n}{}' T^{Q_n} f \tag{5.17}$$

where the primed sum is over all ordered partitions of $\{1, \ldots, n\}$.

(b) The derivatives in (5.12) also satisfy the following: if N is nonnegative, $\phi \in \mathscr{D}_{N+n}$, $\psi_1 \in \mathscr{P}_{N+n+1}$ and $f \in C^{N+n}$, $\partial_{\psi_1, \ldots, \psi_n} T_\phi f$ lies in C^N and, for fixed f and $\{\psi_i\}$, is continuous as a function of ϕ from $\mathscr{D}_{N+n}$ into C^N.

(c) The C^N norm of the derivative is a polynomial in $|f|_M$ $(1 \leqslant M \leqslant N+n)$, in $|H_{\psi_i}|_M$ $(1 \leqslant M \leqslant N+n+1)$ and in the variables $1/[1-\alpha_1(\phi)], \alpha_M(\phi)$ $(1 \leqslant M \leqslant N+n)$. [Here $C^0 = C(\Omega)$.]

We present the estimates needed for the second derivative of the Gibbs state:

$$|\partial_\psi T_\phi f|_\infty \leqslant \frac{\|\psi\|_1 |f|_1}{1-\alpha_1(\phi)} \tag{5.18}$$

$$|\partial_\psi T_\phi f|_1 \leqslant \frac{1}{1-\alpha_1(\phi)} \Big\{ |f|_1(|H_\psi|_2 + |H_\psi|_1 |H_\phi|_2) + |H_\psi|_1 \Big[2|f|_2 + \alpha_2\Big(1 + \frac{2}{1-\alpha_1}\Big)|f|_1 \Big] \Big\} \tag{5.19}$$

$$\begin{aligned} |\partial_{\psi_1,\psi_2} T_\phi f|_\infty \leqslant{} & |H^1|_1 |H^2|_1 |f|_1 [1-\alpha_1(\phi)]^{-1} + [1-\alpha_1(\phi)]^{-1} \\ & \Big\{ \|\psi_1\|_1 (|H^2|_2 + |H^2|_1 |H_\phi|_2) |f|_1 (1-\alpha_1)^{-1} \\ & + (1-\alpha_1)^{-1} \|\psi_1\|_1 |H^2|_1 \Big[2|f|_2 + \alpha_2\Big(1 + \frac{2}{1-\alpha_1(\phi)}\Big)|f|_1 \Big] \\ & + 2\|\psi_1\|_1 \|\psi_2|_1 |f|_1 \\ & + \text{the same terms with } \psi_1 \text{ and } \psi_2 \text{ interchanged} \Big\} \end{aligned} \tag{5.20}$$

Proof. The organization of this proof is as follows:

(i) Let N be a nonnegative integer. We show that the sum of the right-hand side of (5.17) lies in C^N when $\phi \in \mathscr{D}_{N+n}$, $\psi_i \in \mathscr{P}_{N+1+n}$ and $f \in C^{N+n}$. Moreover, its C^N norm is bounded as in part (c) of the lemma. We also indicate the evaluation of the bounds in the right-hand side of (5.18)–(5.20).

(ii) Next we informally derive the formulas (5.16) and (5.17) and then prove part (a) for the first derivative, i.e., $n = 1$.

(iii) We prove the whole of part (a), including the continuity in ϕ, for all n. The ϕ continuity of part (b) will then follow directly from the discussion in (i) and the lemma will have been proved.

(i) Let $N \geqslant 0$ and recall (3.2) of Theorem 3.2. This contains the annoying term $E^p(f)$, which we get rid of by noting that $\sum_p E_n^p(f) = n(1-\alpha_1)|f|_n = [(1-\alpha_1)J|f|]_n$, so that we may replace $J|f| + E^p(f)$ in (3.2) by $\alpha_1 J|f|$. Collecting the estimates in (3.2) and (3.7) for $k \leqslant p$ we see that we have

$$|T_{k,p}f|_N \leqslant |f|_N + (K|f|)_N, \qquad p \geqslant k \tag{5.21}$$

where K is zero on or above its diagonal and whose entries are polynomials in $\alpha_1, \ldots, \alpha_N$ and $1/(1-\alpha_1)$.

Now consider T^Q as in (5.16). By expressing the bounds on $T_{k,p}$ as in (5.21), we place the burden of summability of $T^Q f$ onto the $\zeta_{j_i}^{P_i}$. But they rise to the occasion, thanks to (4.27n) of Theorem 4.4.

Assume the hypotheses of part (b) of this lemma. Then it follows that the summand in (5.16) lies in C^N. For, recalling (4.27n), we have (in matrix terminology) that

$$|\zeta_j^P f|_M \leqslant \left(C^P L^{(j)}(f)\right)_M \tag{5.22}$$

where

$$\left(L^{(j)}(f)\right)_M = \sum_{\substack{\Lambda \in V_{m-1} \\ j \notin \Lambda}} D_{j\cup\Lambda}(f) \tag{5.23}$$

and $C_{M,m}^P$ is a polynomial in $|H_{\psi_i}|_l, |H_\varphi|_l$, $1 \leqslant l \leqslant M+1$, $i \in P$. Now note that we may cast (3.30) of Lemma 3.10 in the form

$$F_{j,p;m}(g) \leqslant \left[(\alpha_1 J + K_1)|g|\right]_m \tag{5.24}$$

where K_1 is a matrix with zeroes on and above its diagonal. If we let

$$g = \text{expression to the right of } \zeta_{j_1}^{P_1} \text{ in (5.16)} \tag{5.25}$$

then we have

$$\begin{aligned}
\sum_{j_1 \leqslant j_2} |\text{summand of RHS (5.16)}|_N &\leqslant \sum_{j_1 \leqslant j_2} \left[(I+K)|\zeta_{j_1}^{P_1} g|\right]_N \\
&\leqslant \sum_{j_1 \leqslant j_2} \left[(I+K)C^{P_1}L^{(j_1)}(g)\right]_N \\
&\leqslant \left[(I+K)C^{P_1}F_{j_1,j_2;\,\cdot}(g)\right]_N \\
&\leqslant \left[(I+K)C^{P_1}(\alpha_1 J + K_1)|g|\right]_N
\end{aligned} \tag{5.26}$$

Iterating this procedure and noting that $\sum_i |P_i| = n$, it is clear that we have, under the hypotheses of part (b) of the current lemma, that the series for $T^Q f$ is absolutely summable in the Nth seminorm. Thus the right-hand side of (5.16) is in C^N itself.

We omit here the evaluation of the estimates (5.18)–(5.20) [the computation of (5.20) was simplified by setting $D_k(\cdot) \leqslant 2|\cdot|_\infty$ at a convenient point].

(ii) Next we show that the sum (5.17) actually represents the derivative. We do this by induction on the number of derivatives. That the right-hand side of (5.17) is informally the derivative with respect to ψ_n of the (informal) derivative $\partial_{\psi_1,\ldots,\psi_{n-1}}T_\phi f$ is simply a question of counting ordered partitions of $[1,n]$ in terms of those of $[1,n-1]$, as in Section 4.

Following Gross[3] let us show that (5.13) holds true, i.e., that the first $C(\Omega)$ derivative exists (for $\phi \in \mathscr{D}_1$, $\psi \in \mathscr{P}_2$ and $f \in C^1$). We use a standard theorem on interchanging limits and differentiation. Let g be a cylinder function. Set $\Lambda_\phi(f) =$ RHS (5.13). It is clear that $\Lambda_\phi(g)$ is a finite sum and is the derivative of $T_\phi g$ (which is a finite product of operators acting on g) by the product rule and the differentiability of the ζ's as in Section 4.1. Moreover, by Theorem 4.4, $\phi \to A_\phi(g)$ is continuous from $\mathscr{D}_1$ into $C(\Omega)$.

Now suppose g_k are cylinder functions converging to f in $C(\Omega)$ and C^1 norms. Fix ϕ and r such that $\alpha_1(\phi) < r < 1$. Recall that definition (5.10) of B_r^1. B_r^1 is open by the continuity of α_1 so that there exists a $\delta > 0$ such that $\phi + v\psi \in B_r^1$ whenever $|v| \leqslant \delta$ and such that $A_{\phi+v\psi}g_k$ converge to $A_{\phi+v\psi}f$ in $C(\Omega)$ norm, uniformly in $v \in [-\delta,\delta]$. This last fact follows from the estimate (5.18).

Thus $A_\phi f$ is indeed the derivative of $T_\phi f$. Moreover, $\phi \to \partial_\psi T_\phi f$ is continuous on $\mathscr{D}_1$. To see this, note that by the bound on $|A_\phi g|_\infty$ (in this case, $[1-\alpha_1(\phi)]^{-1}|g|_1\|\psi\|_1$), $|A_\phi g_n - A_\phi f|_\infty$ converges to zero uniformly in ϕ on B_r^1, $0 < r < 1$. Since the $A_\phi g_n$ are continuous on B_r^1, so is $A_\phi f$.

(iii) Now let n be a natural number, $f \in C^n$, $\phi \in \mathscr{D}_n$, $\psi_i \in \mathscr{P}_{n+1}$ for $1 \leqslant i \leqslant n$. Make the induction hypothesis that for some $m \leqslant n$, $\partial_{\psi_1,\ldots,\psi_{m-1}}T_\phi f$ exists, is bounded and continuous, and satisfies (5.17) (with $n \to m$ there). Let $A_\phi^{(n)}f =$ RHS (5.17) and let $\{g_k\}$ be a sequence of cylinder functions tending to f in $C(\Omega)$ and C^n (and therefore in C^m). Then $A_\phi^{(m-1)}g_k$ is a finite sum for each k. By Theorem 4.4 we have existence and ϕ continuity of the derivatives with respect to ψ_m of $\zeta_j^P g$, and continuity of $T_{\phi,k,p}g$ (again, note that continuity in $\mathscr{P}_1$ implies continuity in $\mathscr{P}_{m+1}$). By the finiteness of the sums involves, $A_\phi^{(m)}g_k = \partial_{\psi_m}A_\phi g_k$. Again, $A_\phi^{(m-1)}g_k \to A_\phi^{(m-1)}f$ in $C(\Omega)$, uniformly in $\phi \in B_r^n$ for each r satisfying $\alpha_1(\phi) < r < 1$, $\alpha_2(\phi), \ldots, \alpha_n(\phi) < 1/(1-r)$ and such that $A_{\phi+v\psi_m}^{(m)}g_k \to A_{\phi+v\psi_m}^{(m)}f$ in $C(\Omega)$, uniformly in v (for suitably restricted v). Both of these facts follow from the estimations in (i) above [as expressed in part (c) of the lemma]. Thus we conclude that $A_\phi^{(m)}f$ is the mth derivative. Moreover, as in (i) above, $|A_\phi^{(m)}(g_k - f)|_\infty$ converges to zero uniformly in ϕ for $\phi \in B_r^N$ and any $r \in (0,1)$, so that continuity in ϕ of the derivatives is assured. By the results of (ii) above, induction is complete and we have proved part (a) and once we have established the remaining portion of part (b) we will have proved the lemma.

What remains is to show that the $C(\Omega)$ derivatives are continuous from $\mathscr{D}_{N+n}$ into C^N. Using cylinder functions as in the above arguments, for example, the estimates in part (c) [already derived in (i) above] justify this conclusion.

5.5. Derivatives of T_ϕ as Operators on C^N

Let π be the canonical map taking C^N to $\tilde{C}^N = C^N/\text{constants}$. $\tilde{C}^N$ is a Banach space in the norm $\sum_{M=1}^{N} |\cdot|_M$. An operator H on C^N which annihilates constants is then uniquely identified with an operator $\tilde{H}$ on $\tilde{C}^N$ through $\tilde{H} \cdot \pi = \pi \cdot H$. We will drop the tilde in what follows. In particular, if $H: C^N \to C(\Omega)$ annihilates constants, we can think of it as an operator from $\tilde{C}^N$ to $C(\Omega)$ through $(H[f])(s) = Hf(s)$.

Since $\tilde{C}^N$ is a Banach space we can define the notion of Gateaux derivatives of functions into $\tilde{C}^N$ and the question arises as to whether $\partial_\psi T_\phi(f)$, which annihilates constants, is in the $\tilde{C}^N$ sense the derivative of some curve in $\tilde{C}^N$. The obvious candidate for its antiderivative is $u \to (T_{\phi+u\psi} - T_\phi)[f]$. The question becomes: is

$$\lim_{u\to 0} \sum_{M=1}^{N} \left| \frac{1}{u}(T_{\phi+u\psi}f - T_\phi f) - \partial_\psi T_\phi f \right|_M = 0 \tag{5.27}$$

for each $f \in C^N$, if $\partial_\psi T_\phi f$ is the already defined $C(\Omega)$ derivative? The answer is yes and may be seen as follows:

Suppose $\phi \in \mathscr{D}_{N+1}$, $\psi \in \mathscr{P}_{N+2}$ and $f \in C^{N+1}$. Then for $M \leqslant N$,

$$\begin{aligned} \left| \frac{1}{u}(T_{\phi+u\psi}f - T_\phi f) - \partial_\psi T_\phi f \right|_M &= \left| \frac{1}{u}\int_0^u dv\, \partial_\psi T_{\phi+v\psi} f - \partial_\psi T_\phi f) \right|_M \\ &\leqslant \frac{1}{|u|}\int_0^u dv\, |\partial_\psi T_{\phi+v\psi} f - \partial_\psi T_\phi f|_M \end{aligned} \tag{5.28}$$

The first step uses the fundamental theorem of the calculus and the second uses Minkowski's inequality: the integrand is integrable by continuity as in Lemma 5.4(b). But by this continuity, we have that the limit as $u \to 0$ is zero (use the other part of the fundamental theorem).

A similar result holds for the derivatives of $\zeta_j^\phi f$ and we see that all the results on existence, boundedness, and continuity of derivatives continue to hold in the setting of $\tilde{C}^N$.

5.6. Lemma. Existence and Continuity of $(I - T_\phi)^{-1}$, and Its Derivatives, on C^N

(a) If $\phi \in \mathscr{D}_N$, the operator $I - T_\phi$ on $\tilde{C}^N$ has a bounded inverse, $S_\phi = (I - T_\phi)^{-1}$. For each fixed $F \in \tilde{C}^N$, $\phi \to S_\phi F : \mathscr{D}_N \to \tilde{C}^N$ is continuous. We have in particular the bounds

$$|S_\phi[f]|_1 \leqslant \frac{|f|_1}{1 - \alpha_1(\phi)} \tag{5.29}$$

$$|S_\phi[f]|_2 \leqslant \frac{|f|_2}{1 - \alpha_1(\phi)} + \alpha_2(\phi)\frac{|f|_1}{[1 - \alpha_1(\phi)]^3} \tag{5.30}$$

(b) If $\phi \in \mathscr{D}_{N+1}$, $\psi \in \mathscr{P}_{N+2}$ and $F \in \tilde{C}^{N+1}$, the function $u \to S_{\varphi + u\psi}F : R^1 \to \tilde{C}^N$ is differentiable at $u = 0$, with derivative

$$(\partial_\psi S)_\phi F = S_\phi \cdot (\partial_\psi T)_\phi \cdot S_\phi F \tag{5.31}$$

where $\partial_\psi T_\phi$ is interpreted as an operator on $\tilde{C}^{N+1}$. [By part (a), $\phi \to \partial_\psi S_\phi F : \mathscr{D}_{N+1} \to \tilde{C}^N$ is continuous.]

Proof. (a) Consider first the sum $\sum_{k=0}^{\infty} |T^k f|_N$. By (3.5) we have

$$|Tf| \leqslant \left(\alpha_1 I + \overline{K}\right)|f|, \qquad f \in C^N, \quad \phi \in D_N \tag{5.32}$$

where the $N \times N$ matrix $\overline{K}$ is zero on and above its diagonal. It follows that

$$\sum_{k=0}^{\infty} |T^k f|_N \leqslant \frac{1}{1-\alpha_1}\left[\exp\left(\frac{1}{1-\alpha_1}\overline{K}\right)|f|\right]_N$$

$$= \frac{1}{1-\alpha}\sum_{m=0}^{N-1}\frac{1}{(1-\alpha_1)^m}\left[\overline{K}^m|f|\right]_N \tag{5.33}$$

Since the entries of K are polynomials in $(1-\alpha_1)^{-1}$, $\alpha_2, \ldots, \alpha_N$, the sum converges uniformly in $\phi \in B_r^N$, $0 < r < 1$, so that it is continuous in ϕ on $\mathscr{D}_N$.

Thus we may define $S_\phi = (I - T_\phi)^{-1}$ by

$$S_\phi[f] = \sum_{k=0}^{\infty} \pi\left(T_\phi^k f\right), \qquad \phi \in \mathscr{D}_N, \quad f \in C^N \tag{5.34}$$

as a linear, bounded, and continuous operator on $\tilde{C}^N$.

When $N = 1$, $K = 0$ so that we have (5.29). When $N = 2$,

$$K = \begin{bmatrix} 0 & 0 \\ \dfrac{\alpha_2}{1-\alpha_1} & 0 \end{bmatrix} \tag{5.35}$$

so that we have (5.30).

(b) Here we take direct advantage of the fact that the alleged derivatives in the right-hand side of (5.31) is already well defined and continuous. Let $f \in F$ so that $f \in C^{N+1}$ and put

$$s_n(\phi) = \sum_{k=0}^{n} \pi\left(T_\phi^k f\right) \tag{5.36}$$

Then by part (a), $\tilde{C}^N - \lim_{n\to\infty} s_n(\phi) = S_\phi F$. It suffices to show that (i) the $\tilde{C}^M$ derivatives $(1 \leqslant M \leqslant N)$ $s_n'(\phi) = (d/du)s_n(\phi + u\psi)|_{n=0}$ exist and are continuous from $\mathscr{D}_{N+1}$ to $\tilde{C}^N$; (ii) $s_n'(\phi)$ converge to $S_\phi \cdot \partial_\psi T_\phi \cdot S_\phi F$ on $\mathscr{D}_{N+1}$ and for each r in $(0,1)$, $\{s_n'(\phi)\}_{n,\phi}$ is uniformly bounded for $1 \leqslant n \leqslant \infty$ and $\phi \in B_r^N$. For then, since $v \to S_{\phi+v\psi} \cdot \partial_\psi T_{\phi+v\psi} \cdot S_{\phi+v\psi}F : R^1 \to \tilde{C}^N$ is continuous (for small v) by part (a) of this lemma and Lemma 5.4, we have

$$S_{\phi+u\psi}F - S_\phi F = \lim_{n\to\infty}\left[s_n(\phi+u\psi) - s_n(\phi)\right]$$
$$= \lim_{n\to\infty}\int_0^u dv\, s_n'(\phi+v\psi)$$
$$= \int_0^u dv\, S_{\phi+v\psi}\cdot \partial_\psi T_{\phi+v\psi}\cdot S_{\phi+u\psi}F \tag{5.37}$$

The second equality follows from the fundamental theorem and (i) above and third, by means of Lebesgue's dominated convergence theorem, from (ii). Finally, the continuity of the last integrand above and another application the fundamental theorem yield the result (5.31).

(i) We have, from (5.36),

$$\frac{S_n(\phi+u\psi)-s_n(\phi)}{u} = \sum_{k=1}^{n}\pi\left[\sum_{j=0}^{k-1} T_{\phi+u\psi}^{k-j-1}\left(\frac{T_{\phi+u\psi}-T_\phi}{u}\right)T_\phi^j f\right] \tag{5.38}$$

For each summand, consider

$$\left|\frac{1}{u}T_{\phi+u\psi}^{k-j-1}(T_{\phi+u\psi}-T_\phi)T_\phi^j f - T_\phi^{k-j-1}\partial_\psi T_\phi\cdot T_\phi^j f\right|_N$$
$$\leqslant \left|T_{\phi+u\psi}^{k-j-1}\left[\frac{1}{u}(T_{\phi+u\psi}-T_\phi)-\partial_\psi T_\phi\right]T_\phi^j f\right|_N$$
$$+\left|\left(T_{\phi+u\psi}^{k-j-1}-T_\phi^{k-j-1}\right)\partial_\psi T_\phi\cdot T_\phi^j f\right|_N \tag{5.39}$$

and consider its limit as u tends to zero. Now, in the last term, $\partial_\psi T_\phi\cdot T_\phi^j f \in C^N$ (Lemmas 5.3 and 5.4). Moreover, T_ϕ (and T_ϕ^{k-j-1}) is, as an operator on C^N, strongly continuous for $\phi\in\mathscr{D}_N$ (Lemma 5.3). Since the injection $\mathscr{D}_{N+1}\to\mathscr{D}_N$ is itself continuous, T_ϕ strongly continuous on $\mathscr{D}_{N+1}$, so that the last term tends to zero.

Let $r<1$, $\phi\in B_r^N$ and $g=T_\phi^j f$. Then $g\in C^{N+1}$ and the second last term of (5.39) is majorized by

$$[C(r)]\sum_{M=1}^{N}\left|\frac{1}{u}(T_{\phi+u\psi}g - T_\phi g)-\partial_\psi T_\phi g\right|_M \tag{5.40}$$

where $C(r)<\infty$ is independent of u (by the bounds on T_ϕ, Lemma 5.3). By the remarks of the last section (5.5), this goes to zero with u and we have that for each n,

$$s_n'(\phi) = \sum_{k=1}^{n}\pi\left(\sum_{j=0}^{k-1}T_\phi^{k-j-1}\cdot\partial_\psi T_\phi\cdot T_\phi^j f\right) \tag{5.41}$$

exists and is continuous from $\mathscr{D}_{N+1}$ to $\tilde{C}^N$. The existence and continuity of these derivatives for the Mth norms, $1\leqslant M\leqslant N$, follows in the same manner.

(ii) We have from (5.41) that

$$|s_n'(\phi)|_N \leqslant \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} |\pi(T_\phi^m \cdot \partial_\psi T_\phi \cdot T_\phi^j f)|_N \tag{5.42}$$

By (5.33) and Lemma 5.4 it is clear that $|s_n'(\phi)|_N$ is uniformly bounded for all n and all $\phi \in B_r^N$, for each $r < 1$. The same is true for $|s_n'(\phi)|_M$, $1 \leqslant M \leqslant N$. Finally, it is clear that $s_n'(\phi)$ converge to $S_\phi \cdot \partial_\psi T_\phi \cdot S_\phi[f]$ in $\tilde{C}^N$ and we are done.

5.7. Proof of Theorem 5.1

We will use induction on the number of derivatives. We follow Ref. 3 in establishing the theorem for the first derivative. Let $f \in C^1$, $\phi \in \mathscr{D}_1$ and $\psi \in \mathscr{P}_2$. Vasershtein's formula says that

$$\sigma_\phi(f) = C^0 - \lim_{n\to\infty} T_\phi^n f \tag{5.43}$$

where $C^0 = C(\Omega)$. By arguments similar to those used in proving Lemma 5.6(b), it is clear that

$$(\partial_\psi T^n)_\phi f = \sum_{j=0}^{n-1} T_\phi^{n-j-1} \cdot \partial_\psi T_\phi \cdot T_\phi^j f \tag{5.44}$$

where now the derivatives are taken in the C^0 sense. Also, this derivative is continuous in ϕ for ϕ in $\mathscr{D}_1$. Now $|T_\phi g|_\infty \leqslant |g|_\infty$ and $|T_\phi f|_1 \leqslant \alpha_1(\phi)|g|_1$ for $\phi \in \mathscr{D}_1$, so that, using (5.18),

$$|(\partial_\psi T^n)_\phi f|_\infty \leqslant \sum_{j=0}^{\infty} \frac{\|\psi\|_1 |T^j f|_1}{1-\alpha_1(\phi)} \leqslant \frac{\|\psi\|_1 |f|_1}{[1-\alpha_1(\phi)]^2} \tag{5.45}$$

Thus the series in (5.44) is absolutely convergent. Put

$$C_{n,j} = \begin{cases} T_\phi^{n-j-1} \partial_\psi T_\phi T_\phi^j f, & 0 \leqslant j \leqslant n-1 \\ 0, & j \geqslant n \end{cases} \tag{5.46}$$

Then $(\partial_\psi T^n)_\phi f = \sum_{j=0}^\infty C_{n,j}$. Since this series is absolutely convergent, uniformly in n, and since by (5.43), $\lim_{n\to\infty} C_{n,j} = \sigma_\phi(\partial_\psi T_\phi T_\phi^j f)$, we have

$$\lim_{n\to\infty} (\partial_\psi T^n)_\phi f = \sum_{j=0}^{\infty} \sigma_\phi(\partial_\psi T_\phi T_\phi^j f) = \sigma_\phi(\partial_\psi T_\phi S_\phi[f]) \tag{5.47}$$

where we have used the convergence of $\sum T^j$ and the boundedness and linearity of $\sigma_\phi \cdot \partial_\psi T_\phi$ on $\tilde{C}^1$.

Now $\phi \to \partial_\psi T_\phi \cdot S_\phi[f] : \mathscr{D}_1 \to C^0$ is continuous. But so is $\phi \to \sigma_\phi(g) : \mathscr{D}_1 \to R^1$ for each $g \in C(\Omega)$. We may see this from Vasershtein's proof of Dobrushin's uniqueness theorem (Ref. 23, Section 3.12), from which we may extract the fact that for $g \in C^1$,

$$|T_\phi^n g - \sigma_\phi(g)|_\infty \leqslant \sup(T_\phi^n g) - \inf(T_\phi^n g) \leqslant |T_\phi^n g|_1 \leqslant \alpha_1^n(\phi)|g|_1 \quad (5.48)$$

so that the convergence of $T_\phi^n g$ to $\sigma_\phi(g)$ is uniform on each B_r^1. Thus $\phi \to \sigma_\phi(g)$ is continuous on $\mathscr{D}_1$ for g in the $C(\Omega)$-dense set C^1 and hence it is continuous for all g in $C(\Omega)$.

But this means that $\phi \to \sigma_\phi(\partial_\psi T_\phi \cdot S_\phi[f])$ is continuous from $\mathscr{D}_1$ into the reals. We may now use the fundamental theorem of the calculus and the dominated convergence theorem to see that, for u small enough,

$$\begin{aligned}\sigma_{\phi+u\psi}(f) - \sigma_\phi(f) &= \lim_{n\to\infty} \int_0^u dv\,(\partial_\psi T^n)_{\phi+v\psi}(f) \\ &= \int_0^u dv\, \sigma_{\phi+v\psi}(\partial_\psi T_{\phi+v\psi} \cdot S_{\phi+v\psi}[f]) \qquad (5.49)\end{aligned}$$

By the continuity of the integrand, we have again from the fundamental theorem that $\sigma_\phi(f)$ is indeed differentiable, with the expected derivative. Note that (5.5) is fully justified.

Now proceed with the induction hypothesis that the theorem is true for some N. Let $\phi \in \mathscr{D}_{N+1}$, $\psi \in \mathscr{P}_{N+2}$ and $f \in C^{N+1}$. By hypothesis, the Nth derivative in directions $\psi_1, \ldots, \psi_N$ exists, is continuous, and is given by (5.2) with $n = N$ (we consider the ψ_i to lie in $\mathscr{P}_{N+2}$). (5.2) is of the form

$$\partial_{[\psi_i]_1^N}\sigma_\phi(f) = \sum_{Q_N}{}' \sigma_\phi(f_{Q_N}) \qquad (5.50)$$

where, with Q an ordered partition of $\{1, \ldots, N\}$,

$$f_Q(\phi) = T_1'(\phi)T_2'(\phi)\cdots T_m'(\phi)[f], \qquad m = |Q| \qquad (5.51)$$

$$T_i'(\phi)G = (\partial_{\{\psi_j\}_{j\in P_i}} T)_\phi \cdot S_\phi G \qquad (G \text{ in some } \tilde{C}^n) \qquad (5.52)$$

Thus we have that

$$\begin{aligned}&\frac{1}{u}[f_Q(\phi+u\psi) - f_Q(\phi)] \\ &= \frac{1}{u}\sum_{i=1}^m T_1'(\phi+u\psi)\cdots T_{i-1}'(\phi+u\psi)[T_i'(\phi+u\psi) - T_i'(\phi)] \\ &\quad \times T_{i+1}'(\phi)\cdots T_m'(\phi)[f] \qquad (5.53)\end{aligned}$$

(with suitable adjustments for $i = 1, m$).

From Lemmas 5.4 and 5.6 and Remark 5.5 we have that

$$T_l'(\phi) : \tilde{C}^{N+1-\sum_{j=l+1}^m |P_j|} \to \tilde{C}^{N+1-\sum_{j=l}^m |P_j|} \qquad (5.54)$$

boundedly, $i + 1 \leqslant l \leqslant m$;

$$\frac{1}{u}[T_i'(\phi+u\psi) - T_i'(\phi)]T_{i+1}'(\phi)\cdots T_m'(\phi)[f] \qquad (5.55)$$

has the limit

$$\begin{aligned}(\partial_\psi T_i')(\phi) \cdot T_{i+1}'(\phi) \cdots T_m'(\phi)[f] \qquad &\text{on } \tilde{C}^{N-\sum_{j=1}^{m}|P_j|} \\ T_l'(\phi + u\psi) : \tilde{C}^{N-\sum_{l+1}^{m}|P_j|} \to &\tilde{C}^{N-\sum_l^m|P_j|}\end{aligned} \tag{5.56}$$

boundedly and is continuous at $u = 0$. Hence we may let $u \to 0$ in (5.53) to obtain the result that the $C(\Omega)$ derivative of f_Q exists and is given by

$$\partial_\psi f_Q(\phi) = \sum_{i=1}^{m} T_1'(\phi) \cdots T_{i-1}'(\phi) \cdot (\partial_\psi T_i')(\phi) T_{i+1}'(\phi) \cdots T_m'(\phi)[f] \tag{5.57}$$

Also, note that, by similar arguments

$$\partial_\psi T_i'(\phi)G = \partial_{(\{\psi_j\}_{j\in p})\cup\{\psi\}} T_\phi \cdot S_\phi G + \partial_{\{\psi_j\}_{j\in P}} T_\phi \cdot S_\phi \cdot \partial_\psi T_\phi \cdot S_\phi G \tag{5.58}$$

from (5.31). Now

$$\begin{aligned}\sigma_{\phi+u\psi}[f_Q(\phi - u\psi)] - \sigma_\phi[f_Q(\phi)] = \{&\sigma_{\phi+u\psi}[f_Q(\phi)] - \sigma_\phi[_Q(\phi)]\} \\ &+ \sigma_{\phi+u\psi}[f_Q(\phi + u\psi) - f_Q(\phi)]\end{aligned} \tag{5.59}$$

In the last term, $\sigma_{\phi+u\psi}(\cdot)$ is continuous in u [since $\phi \to \sigma_\phi(g)$ is continuous on $\mathscr{D}_1$ for each $g \in C(\Omega)$, it is also so on $\mathscr{D}_{N+1}$]. Hence we conclude, using (5.50), (5.57), and (5.58) that the $(N+1)$th derivative exists and equals the right-hand side of (5.2) with $n = N+1$, $\psi = \psi_{N+1}$. Continuity in ϕ follows from that of the various constituents, and our induction is complete. The boundedness remarks in the second paragraph of Section 5.1 follow from Lemmas 5.4 and 5.6 and the proof of the theorem is complete.

The bound (5.8) is obtained by applying (5.18), (5.19), (5.20), (5.29), and (5.30) to (5.6).

5.8. Corollary: Differentiability of the Pressure

Let $L = Z^\nu$. If $N \geqslant 2$, the pressure is N times continuously Gateaux differentiable in $\hat{\mathscr{P}}_N$ directions on $\hat{\mathscr{D}}_{N-1}$ (the caret denoting translational invariance). The Nth derivative at $\phi \in \hat{\mathscr{D}}_{N-1}$ in the $\hat{\mathscr{P}}_N$ directions $\psi_1, \ldots, \psi_N$ is bounded in absolute value by a polynomial in

$$\frac{1}{1-\alpha_1(\phi)}, \alpha_M(\phi)\ (1 \leqslant M \leqslant N-1); \quad \|\psi_N\|_M\ (1 \leqslant M \leqslant N-1)$$

$$|H_\phi|_M\ (1 \leqslant M \leqslant N); \quad \text{and} \quad |H_{\psi_i}|_M\ (1 \leqslant M \leqslant N,\ 1 \leqslant i \leqslant N-1)$$

The Nth derivative of the pressure may be expressed as the series (5.2) with

$n = N - 1$ and $f = A_{\psi_N}$ there, where, with 0 the origin of L,

$$A_\psi = - \sum_{\substack{\Lambda \subset L \\ 0 \in \Lambda}} |\Lambda|^{-1} \psi(\cdot \mid \Lambda) \tag{5.60}$$

Proof. We mentioned in Section 1.1 that on $\hat{\mathscr{D}}_1$ the pressure P is continuously Gateaux differentiable in $\hat{\mathscr{P}}_2$ directions ψ, with

$$\partial_\psi P(\phi) = \sigma_\phi(A_\psi) \tag{5.61}$$

The corollary follows from Theorem 5.1 upon observing that, as in Section 2.5, $A_\psi \in C^M$ for $\psi \in P_M$.

We have, for the first derivative, from (5.61) that

$$|\partial_\psi P(\phi)| \leqslant \|\psi\|_1 \tag{5.62}$$

The second and third derivatives may also be estimated by means of (5.7), (5.8), (5.61), and (2.21). For $N = 2$, for example, we have Gross' result that

$$|\partial_{\psi,\psi_1} P(\phi)| \leqslant \frac{2\|\psi\|_1 \|\psi\|_1}{(1 - \alpha_1(\phi))^2} \tag{5.63}$$

For $N = 3$ we may use (5.8): here $|A_\psi|_2 \leqslant 2\|\psi\|_2$.

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